IJARSCT

ISSN (Online) 2581-9429

International Journal of Advanced Research in Science, Communication and Technology (IJARSCT)

Impact Factor: 7.301

Volume 3, Issue 1, February 2023

Application of the Aboodh Iterative Method to Fractional Partial Differential Equations

I. A. Almardy[1], R. Farah[2,4], M. Y. Ahmed[2,3], M. A. Alkeer[1] , M. A. Mohammed[1] and A. K. Osman[1]

Department of MIS-Stat. and Quant. Methods Unit

Faculty of Business and Economics, University of Qassim, Buraidah, KSA, Buraidah[1]

Department of Mathematics, Faculty of Science, University of Qassim, Buraidah, KSA, Buraidah[2]

Department of Mathematics, Faculty of Education, University of Nyala, Sudan[3]

Department of Mathematics, Faculty of Science &Technology, Omdurman Islamic University, Khartoum, Sudan[4]

Corresponding Author's Email address: i.Abdallah@qu.edu.sa[1]

Abstract: *In this article, we present an iterative transformation method for solving fractional partial differential equations that combines the Aboodh transform and iterative methods. By this iterative transformation method, numerical solutions in the form of series are obtained. When we apply this method to the fractional linear Klein–Gordon equation, we find that it yields the same results, just like the Homotopy perturbation method. The procedures and results of this method for solving the new generalized fractional Hirota–Satsuma coupled KdV equation are given in the paper.*

Keywords: Aboodh transform; Iterative method; Fractional differential equation

I. INTRODUCTION

In recent decades, fractional-order partial differential equations have been widely used and developed in physics, engineering, and fluid mechanics. Compared with integer-order partial differential equations, they are more suitable to portray complex phenomena and processes. Therefore, the method to solve fractional partial differential equations is also a relatively important problem. Now, there are methods to solve fractional-order partial differential equations. For example, in [13], the finite-difference methods, the Galerkin finiteelement methods, and the spectral methods to solve fractional-order partial differential equations are mentioned; Gepreel uses the Homotopy perturbation method to obtain the solution of the fractional Klein–Gordon equation in [14]; Khalid used the Aboodh transform method to solve the equations [12]; in [4], Ziane used the fractional Aboodh variational iteration method to solve the equations; Jafari introduced the Iterative Laplace transform method in [2]; Aboodh used the Sumudutrans form of the variational iteration method to solve linear homogeneous partial differential equations; Thabet [9] introduced a new analytic method to solve partial differential equations with fractional order, and El-Rashidy used the method to obtain new traveling-wave solutions of the equations. Hosseini used the modified Kudryashov method to obtain exact solutions of the coupled sineGordon equations. Mohammad Tamsir employed a semianalytical approach to obtain the approximate analytical solution of the Klein–Gordon equations; The Klein–Gordon equation is a crucial equation in the study of relativistic quantum mechanics. Many authors have solved the generalized Hirota–Satsuma coupled KdV equation utilizing various equations , including the homotopy analysis approach . Thisis a significant class of equations in mathematics and physics. In this study, we employ the Aboodh transform in conjunction with an iterative approach to generate approximations to partial differential equations with fractional order. The results demonstrate the method's validity, further, it also may be applied to other fractional-order partial differential equations.

Basic Definition

Definition 1 The fractional Riemann–Liouville of operator D^p is as follows

$$D^p w(x) = \begin{cases} \dfrac{\partial^m w(x)}{\partial x^m} . p = m \\ \dfrac{\partial}{\Gamma(m-p)\partial x^m}\displaystyle\int_0^x \dfrac{w(x)}{(x-\varepsilon)^{p-m+1}} d\varepsilon . m-1 < p < m \end{cases} \quad (1)$$

Where $m \in z^+ . p \in R^+$ $where$ $0 < p \leq 1$

$$D^p w(x) = \frac{1}{\Gamma(p)} \int_0^x \frac{w(x)}{(x-\varepsilon)^{1-p}} d\varepsilon.$$

Definition 2 The Riemann–Liouville integral operator with fractional order is defined asfollows

$$I^p w(x) = \frac{1}{\Gamma(p)} \int_0^x (x-\varepsilon)^{p-1} w(\varepsilon) d\varepsilon . p > 0. \varepsilon > 0 \ (2)$$

Definition 3 The Caputo fractional derivative of $w(x)$is defined as follows, m ∈ N

$$CD^p w(x) = \begin{cases} \dfrac{\partial^m w(x)}{\partial x^m} . p = m \\ I^{m-p} \left[\dfrac{\partial^m w(x)}{\partial x^m} \right] . m-1 < p < m \ (3) \end{cases}$$

Definition 4 The Aboodh [1, 10] transform of $w(x)$is defined as follows

$$A[w(x)] = K(v) = \frac{1}{v} \int_0^\infty w(x) e^{-vx} \, dx . x \geq 0 . k_1 \leq v \leq k_1 \ (4)$$

Definition 5 The fractional Caputo operator of the Aboodh transform is [24]

$$A\left[D_x^\alpha w(x)\right] = v^\alpha A[w(x)] - \sum_{k=0}^{m-1} \frac{1}{v^{2-\alpha+k}} w^{(k)}(0). where\ m-1 < \alpha < m \ (5)$$

II. METHODOLOGY OF THE ABOODH TRANSFORM ITERATIVE METHOD

To briefly describe this equation in detail, we consider the following fractional partial differential equations

$$\frac{\partial^\alpha w(x.t)}{\partial t^\alpha} = M\big(w_1(x.t). w_2(x.t). \dots \dots . w_n(x.t)\big) + N\big(w_1(x.t). w_2(x.t). \dots \dots . w_n(x.t)\big) \ (6)$$

$where\ M\ and\ N$ are the nonlinear and linear operators from Banach space $B \rightarrow B$, respectively, $\alpha \in R^{+}$, $m-1 < \alpha \leq m . m = 0.1. \dots . n$

subject to the initial condition

$$\frac{\partial^k w(x.0)}{\partial t^k} = w_k(x.0). k = 0.1. \dots \dots . m-1. m \in N$$

Then, the Aboodh transformation acts simultaneously on both sides of the equation, and we obtain

$$A\left[\frac{\partial^k w(x.0)}{\partial t^k}\right] = A\big[M\big(w_1(x.t). w_2(x.t). \dots \dots .. w_n(x.t)\big) + N\big(w_1(x.t). w_2(x.t). \dots \dots .. w_n(x.t)\big)\big]. (8)$$

Hence

$$v^k A[w(x.t)] - \sum_{n=0}^{m-1} \frac{1}{v^{2-k+n}} w^{(n)}(x.0) =$$

$$A\big[M\big(w_1(x.t). w_2(x.t). \dots \dots .. w_n(x.t)\big) + N\big(w_1(x.t). w_2(x.t). \dots \dots . w_n(x.t)\big)\big](9)$$

Through the use of the inverse Aboodh transform, we obtain

$$w(x.t) = A^{-1}\left[\sum_{n=0}^{m-1} \frac{1}{v^{2+n}} w^{(n)}(x.0)\right]$$

$$+ A^{-1}\left[v^{-k} A\big[M\big(w_1(x.t). w_2(x.t). \dots \dots . w_n(x.t)\big) + N\big(w_1(x.t). w_2(x.t). \dots \dots . w_n(x.t)\big)\big]\right] \ (10)$$

The following iterative method is utilized

$$w(x.t) = \sum_{i=0}^\infty w_i(x.t)(11)$$

The nonlinear operator M can be decomposed into

$$M\big(w_1(x.t).w_2(x.t).\dots\dots.w_n(x.t)\big) = M\big(w_{10}(x.t).w_{20}(x.t).\dots\dots.w_{n0}(x.t)\big) + \sum_{m=0}^{\infty}\left[M\left(\sum_{i=0}^{m} w_{1i}(x.t).\sum_{i=0}^{m} w_{2i}(x.t).\dots\dots.\sum_{i=0}^{m} w_{ni}(x.t)\right) - M\left(\sum_{i=0}^{m-1} w_{1i}(x.t).\sum_{i=0}^{m-1} w_{2i}(x.t).\dots\dots.\sum_{i=0}^{m-1} w_{ni}(x.t)\right)\right] \quad (12)$$

Then, we can obtain

$$\sum_{i=0}^{\infty} w_i(x.t) = A^{-1}\left[\sum_{n=0}^{m-1}\frac{1}{v^{2+n}}w^{(n)}(x.0)\right] + A^{-1}v^{-k}\left[A\left[M\big(w_{10}(x.t).w_{20}(x.t).\dots\dots.w_{n0}(x.t)\big) + N\big(w_{10}(x.t).w_{20}(x.t).\dots\dots.w_{n0}(x.t)\big)\right]\right] + A^{-1}\left[v^{-k}A\left[\sum_{m=0}^{\infty}\left[M\left(\sum_{i=0}^{m} w_{1i}(x.t).\sum_{i=0}^{m} w_{2i}(x.t).\dots\dots.\sum_{i=0}^{m} w_{ni}(x.t)\right) - M\left(\sum_{i=0}^{m-1} w_{1i}(x.t).\sum_{i=0}^{m-1} w_{2i}(x.t).\dots\dots.\sum_{i=0}^{m-1} w_{ni}(x.t)\right)\right]\right]\right] + A^{-1}\left[v^{-k}A\left[\sum_{m=0}^{\infty}\left[N\left(\sum_{i=0}^{m} w_{1i}(x.t).\sum_{i=0}^{m} w_{2i}(x.t).\dots\dots.\sum_{i=0}^{m} w_{ni}(x.t)\right) - N\left(\sum_{i=0}^{m-1} w_{1i}(x.t).\sum_{i=0}^{m-1} w_{2i}(x.t).\dots\dots.\sum_{i=0}^{m-1} w_{ni}(x.t)\right)\right]\right]\right] \quad (13)$$

We make the following settings

$$\begin{cases} w_0 = A^{-1}\left[\sum_{n=0}^{m-1}\frac{1}{v^{2+n}}w^{(n)}(x.0)\right]. \\ w_1 = A^{-1}v^{-k}\left[A\begin{bmatrix} M\big(w_{10}(x.t).w_{20}(x.t).\dots\dots.w_{n0}(x.t)\big) \\ +N\big(w_{10}(x.t).w_{20}(x.t).\dots\dots.w_{n0}(x.t)\big)\end{bmatrix}\right]. \\ w_i = A^{-1}v^{-k}\left[A\left[M\big(w_{10}(x.t).w_{20}(x.t).\dots\dots.w_{n0}(x.t)\big) + N\big(w_{10}(x.t).w_{20}(x.t).\dots\dots.w_{n0}(x.t)\big)\right]\right] \\ -M\left(\sum_{i=0}^{m-1} w_{1i}(x.t).\sum_{i=0}^{m-1} w_{2i}(x.t).\dots\dots.\sum_{i=0}^{m-1} w_{ni}(x.t)\right) \\ +A^{-1}\left[v^{-k}A\left[\sum_{m=0}^{\infty}\begin{bmatrix} N\left(\sum_{i=0}^{m} w_{1i}(x.t).\sum_{i=0}^{m} w_{2i}(x.t).\dots\dots.\sum_{i=0}^{m} w_{ni}(x.t)\right) - \\ N\left(\sum_{i=0}^{m-1} w_{1i}(x.t).\sum_{i=0}^{m-1} w_{2i}(x.t).\dots\dots.\sum_{i=0}^{m-1} w_{ni}(x.t)\right)\end{bmatrix}\right]\right] \end{cases} \quad (14)$$

Finally, we obtain the approximate solution of the fractional-order partial differential equation

$$w(x.t) \cong w_0(x.t)+w_1(x.t) + w_2(x.t) + \dots + w_m(x.t).m = 1.2.3.\dots\dots (15)$$

Theorem B is the Banach space, if there exists $0 < K < 1$, $\|w_n\| \le K\|w_n\| \; for \; \forall x \in N$. then the approximate solution $w(x.t)$ converges to A.

Proof Define the sequence A_i, i = 0, 1, ..., n

$$\begin{cases} A_0 = w_0 \\ A_1 = w_0 + w_1 \\ A_2 = w_0 + w_1 + w_2 \\ \dots\dots\dots. \\ A_n = w_0 + w_1 + \dots\dots\dots + w_n \end{cases} \quad (16)$$

and prove that $(A_i)_{i\geq 0}$ is a Cauchy sequence, and we consider

$$\|A_n - A_{n-1}\| \leq \|A_n\| \leq K^n w_0 \ (17)$$

$for\ m > n > 0 \in N. we\ have$

$$\begin{cases} \|A_n - A_m\| = \|A_n - A_{n-1} + A_{n-1} - A_{n-2} + \cdots + A_{m+1} - A_m\| \\ \leq \|A_n - A_{n-1}\| + \|A_{n-1} - A_{n-2}\| + \cdots + \|A_{m+1} - A_m\| \\ \leq (K^n + K^{n-1} + \cdots + K^{m+1})A_0 \\ \leq \left\| \frac{K^{m+1}(1-K^{n-m})}{1-K} \right\| A_0 \end{cases} \quad (18)$$

where w_0 is bounded, and we have

$\lim_{n.m} \|A_n - A_m\| = 0$ (19)

Therefore, the sequence $(A_i)_{i\geq 0}$ is a Cauchy sequence in B, so the solution of Eq. (6) is convergent. The error estimates are as follows:

$$sup \left| w(x.t) - \sum_{i=0}^{m} w_i(x.t) \right| \leq \frac{K^{m+1}}{1-K} \sup|w_0(x.t)| \ (20)$$

Remark Similar proofs can be found in [9].

III. TEST EXAMPLE

Example 1 Consider the linear fractional Klein–Gordon equation [6]

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} - u = 0.0 < \alpha \leq 1 \ (21)$$

subject to the initial condition:

$u(x.0) = 1 + \sin x.$ (22)

The Aboodh transform of the linear fractional Klein–Gordon equation [11] is

$$v^\alpha A[u(x.t)] = \left(\frac{1}{v}\right)^{2-\alpha} u(x.0) + A\left[\frac{\partial^2 u(x.t)}{\partial x^2} + u(x.t)\right] (23)$$

Using the inverse Aboodh transform of the above equation, we obtain

$$u(x.t) = A^{-1}\left[\frac{1}{v^2}\, u(x.0)\right] + A^{-1}\left[v^{-\alpha} A\left[\frac{\partial^2 u(x.t)}{\partial x^2} + u(x.t)\right]\right] (24)$$

then, we use the iterative method above, and we have

$$u_0(x.t) = A^{-1}\left[\frac{1}{v^2}\,(1+sinx)\right] (25)$$

$$u_0(x.t) = 1 + sinx \ (26)$$

$$u_1(x.t) = A^{-1}\left[v^{-\alpha} A\left[\frac{\partial^2 u_0(x.t)}{\partial x^2} + u_0(x.t)\right]\right] (27)$$

$$u_1(x.t) = \frac{t^\alpha}{\Gamma(1+\alpha)} (28)$$

$$u_2(x.t) = v^{-\alpha} A\left[\frac{\partial^2 u_1(x.t) + u_0(x.t)}{\partial x^2} + u_1(x.t) + u_0(x.t)\right] - v^{-\alpha} A\left[\frac{\partial^2 u_0(x.t)}{\partial x^2} + u_0(x.t)\right] (29)$$

$$u_2(x.t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} (30)$$

$$u_3(x.t) = v^{-\alpha}A\left[\frac{\partial^2(u_2(x.t)+u_1(x.t)+u_0(x.t))}{\partial x^2}+u_2(x.t)+u_1(x.t)+u_0(x.t)\right] - v^{-\alpha}A\left[\frac{\partial^2(u_1(x.t)+u_0(x.t)}{\partial x^2}+u_1(x.t)+u_0(x.t)\right] (31)$$

$$u_3(x.t) = \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} (32)$$

…………………………………

$$u_n(x.t) = \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} (33)$$

The result is

$$u(x.t) = 1 + \sin x + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots + \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} (34)$$

When $\alpha = 1.$ the exact solution of the linear fractional Klein–Gordon equation is as follows: $u(x.t) = e^{t} + \sin x$

IV. CONCLUSION

In this article, we use the Aboodh transform with an iterative method to solve fractional partial differential equations. We find that the results using the homotopy perturbation method and the method in this article to the Klein–Gordon problem are the same. We see that the errors were not significant by picking specific values. Therefore, employing the Aboodh transform and the iterative method to solve fractional partial differential equations is effective.

ACKNOWLEDGMENT

The authors extend their appreciation to the Deputyship for Research &Innovation, Ministry of Education and Qassim University, Saudi Arabia for funding this research work through the project number (QU- 6640- 51452).

REFERENCES

[1]. I. A. Almardy , R. A. Farah, M. A. Alkeer , K. S. Aboodh , A. K. Osman , M. A. Mohammed, On the Solution of Integro-Differential Equation Systems by using AboodhTransform. Volume 3, Issue 2, January 2023, 556-563.

[2]. Jafari, H., Nazari, M., Baleanu, D., Khalique, C.M.: A new approach for solving a system of fractional partial differential equations. Comput. Math. Appl. 66(5), 838–843 (2013)

[3]. K.S.Aboodh, R.A.Farah, I.A.Almardy and F.A.Almostafa, some Application of Aboodh Transform to First Order Constant Coefficients Complex equations, International Journal of Mathematics and its Applications , ISSN : 2347- 1557,App.6(1-A)(2018),1-6.

[4]. Applications of Double Aboodh Transform to Boundary Value Problem I. A. Almardy, R. A. Farah,H. Saadouli , K. S. Aboodh 1, A. K. Osman (2023) (IJARSCT) Volume 3 , Issue 1.

[5]. K.S.Aboodh, I.A.Almardy , R.A.Farah,M.Y.Ahmed and R.I.Nuruddeen, On the Application of Aboodh Transform to System of Partial Differential Equations, BEST, IJHAMS Journal, ISSN(P): 2348-0521; ISSN(E): 2454-4728 Volume 10, Issue 2, Dec 2022.UIFUYFYHVHVNOOJIOHIUHG

[6]. Scott, A.C.: A nonlinear Klein-Gordon equation. Am. J. Phys. 37(1), 52–61 (1969)

[7]. Wu, Y., Geng, X., Hu, X., Zhu, S.: A generalized Hirota–Satsuma coupled Korteweg–de Vries equation and Miura transformations. Phys. Lett. A 255(4–6), 259–264 (1999)

[8]. Solution of Partial Integro-Differential Equations by using Aboodh and Double Aboodh Transform Methods, K.S. Aboodh , R.A. Farah, I.A. Almardy and F.A. ALmostafa, Global Journal of Pure and Applied Mathematics. ISSN 0973-1768 Volume 13, Number 8 (2017), pp. 4347-4360 © Research India Publications http://www.ripublication.com

[9]. Thabet, H., Kendre, S., Chalishajar, D.: New analytical technique for solving a system of nonlinear fractional partial differential equations. Mathematics 5(4) (2017)

[10]. K.S.Aboodh, R.A.Farah, I.A.Almardy and F.A.Almostafa, Solution of partial Integro-Differential Equations by using Aboodh and Double Aboodh Transform Methods, Global Journal of pure and Applied Mathematics, ISSN 0973-1768 Volume 13, Number 8 (2017), pp.4347-4360

[11]. K.S.Aboodh,M.Y.Ahmed, R.A.Farah, I.A.Almardy and M.Belkhamsa, New Transform Iterative Method for Solving some Klein-Gordon Equations, (IJARSCT) IIUI, ISSN 1 Volume 2, (2022), pp.118-126. SCOPe Database Article Link: https://sdbindex.com/documents/00000310/00001-85016.pdf

[12]. Solving Delay Differential Equations By Aboodh Transformation Method K. S. Aboodh,1, R. A. Farah, I. A. Almardy& A. K. Osman, International Journal of Applied Mathematics & Statistical Sciences (IJAMSS) ISSN(P): 2319-3972; ISSN(E): 2319-3980 Vol. 7, Issue 2, Feb - Mar 2018; 55 -64.

[13]. Li, C., Chen, A.: Numerical methods for fractional partial differential equations. Int. J. Comput. Math. 95(6–7), 1048–1099 (2018).

[14]. Gepreel, K.A.: The homotopy perturbation method applied to the nonlinear fractional Kolmogorov–Petrovskii–Piskunov equations. Appl. Math. Lett. 24(8), 1428–1434 (2011).