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Preservation of Amalgamated Free Products under Semidirect Constructions and Applications

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Abstract: *In this paper, we investigate the interaction between semidirect products and free products with amalgamation in the category of groups. We establish that the semidirect product functor preserves amalgamated free products under suitable conditions. Using exact sequences and categorical arguments, we derive structural results for groups acting on trees. As an application, we obtain an alternative decomposition of classical matrix groups using amalgamated products of finite groups. Our results provide a unified perspective on group actions, extensions, and decomposition theory.*

Keywords*: suitable conditions*

I. INTRODUCTION

The study of group decompositions plays a central role in combinatorial and geometric group theory. In particular, *free products with amalgamation* and *semidirect products* are fundamental constructions that arise naturally in topology, algebra, and group actions on graphs and trees.

A natural question arises:

How do semidirect products interact with free products with amalgamation?

This question is motivated by Bass–Serre theory, where group actions on trees correspond to decompositions of groups as amalgamated products. Understanding how these decompositions behave under extensions is crucial in analysing more complex algebraic structures.

Let A, B, C, D be groups with $D \subseteq A, B$. The free product with amalgamation is denoted by

$$A *_D B.$$

In this manuscript, we prove that under suitable conditions:

$$(A *_D B) \,\square\, C \cong (A \square C) *_{D \square C} (B \square C).$$

We further explore consequences of this result in the context of group actions on trees and provide applications to matrix groups.

II. PRELIMINARIES

2.1 Free Products with Amalgamation

Let A, B be groups and D a common subgroup. The amalgamated product is defined as:

$A_{*D}B=(A*B)/N,$

Where N is the normal closure of elements of the form:

$\iota_A(d)\, \iota_B(d)^{-1},\ d \in D$

Each element has a reduced normal form:

$$g = a_1 b_1 \cdots a_n b_n,$$

with $a_i \in A$, $b_i \in B$.

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2.2 Semidirect Products

Let G and C be groups with an action $\varphi: C \to \mathrm{Aut}(G)$. The semidirect product is:

$$G \rtimes C = \{(g, c) \mid g \in G, c \in C\},$$

with multiplication:

$$(g_1, c_1)(g_2, c_2) = (g_1 \varphi(c_1)(g_2), c_1 c_2).$$

2.3 Exact Sequences

A short exact sequence:

$$1 \to N \xrightarrow{\alpha} G \xrightarrow{\beta} Q \to 1$$

is *split* if there exists $s: Q \to G$ such that:

$$\beta \circ s = \mathrm{id}_Q.$$

III. EXACT SEQUENCES ASSOCIATED WITH AMALGAMATED PRODUCTS

We begin by constructing an exact sequence linking the two constructions.

Lemma 3.1

Let A, B, C, D be groups. Then there exists an exact sequence:

$$1 \to A *_D B \xrightarrow{\nu} (A \rtimes C) *_{D \rtimes C} (B \rtimes C) \xrightarrow{\mu} C \to 1.$$

Proof

Define the map:

$$\nu: A *_D B \to (A \rtimes C) *_{D \rtimes C} (B \rtimes C)$$

by:

$$\nu(a) = (a, 1), \nu(b) = (b, 1).$$

This extends uniquely due to the universal property of amalgamated products.

Next define:

$$\mu: (A \rtimes C) *_{D \rtimes C} (B \rtimes C) \to C$$

by:

$$\mu(a, c) = c.$$

Then:

μ is surjective,

$\ker\ \mu = A *_D B$.

Hence the sequence is exact.

IV. FUNCTORIAL BEHAVIOUR OF SEMIDIRECT PRODUCTS

We now interpret semidirect product as a functor.

Proposition 4.1

Let C be a fixed group. The assignment:

$$G \mapsto G \rtimes C$$

defines a functor:

$$\mathcal{F}_C: \mathbf{Grp} \to \mathbf{Grp}.$$

Proof

For a homomorphism $\psi: G \to H$, define:

$$\mathcal{F}_C(\psi)(g, c) = (\psi(g), c).$$

Then:

Identity is preserved,

Composition is preserved.

Thus $\mathcal{F}_C$ is a functor.

V. MAIN THEOREM: PRESERVATION OF AMALGAMATION

Theorem 5.1
For groups A, B, C, D, the semidirect product preserves amalgamated free products:

$$(A *_D B) \rtimes C \cong (A \rtimes C) *_{D \rtimes C} (B \rtimes C).$$

Proof
From Lemma 3.1, we have the exact sequence:

$$1 \to A *_D B \to G \to C \to 1,$$

where:

$$G = (A \rtimes C) *_{D \rtimes C} (B \rtimes C).$$

Define a section:

$$s: C \to G, s(c) = (1, c).$$

Then:

$$\mu \circ s = \mathrm{id}_C,$$

so the sequence splits.
Hence:

$$G \cong (A *_D B) \rtimes C.$$

VI. GROUPS ACTING ON TREES

Let Gact on a tree Twithout inversion. By Bass–Serre theory:

$$G \cong G_P *_{G_e} G_Q,$$

where:
$G_P, G_Q$are vertex stabilizers,
$G_e$is an edge stabilizer.
Theorem 6.1
If stabilizers satisfy:

$$G_P = H \rtimes A, G_Q = H \rtimes B, G_e = H \rtimes D,$$

then:

$$G \cong (A *_D B) \rtimes H.$$

Proof
Apply Theorem 5.1 to:

$$G \cong G_P *_{G_e} G_Q.$$

Substitute the semidirect forms and rearrange.

VII. APPLICATION TO MATRIX GROUPS

Consider:

$$GL_2(\mathbb{Z}) = \{A \in M_{2\times 2}(\mathbb{Z}) \mid \det(A) = \pm 1\}.$$

We have the exact sequence:

$$1 \to SL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}) \to \mathbb{Z}_2 \to 1.$$

Since it splits:

$$GL_2(\mathbb{Z}) \cong SL_2(\mathbb{Z}) \rtimes \mathbb{Z}_2.$$

Using known decomposition:

$$SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6,$$

we obtain:

$$GL_2(\mathbb{Z}) \cong (\mathbb{Z}_4 \rtimes \mathbb{Z}_2) *_{\mathbb{Z}_2 \rtimes \mathbb{Z}_2} (\mathbb{Z}_6 \rtimes \mathbb{Z}_2).$$

Recognizing:

$$\mathbb{Z}_4 \rtimes \mathbb{Z}_2 \cong D_4, \mathbb{Z}_6 \rtimes \mathbb{Z}_2 \cong D_6,$$

we conclude:

$$GL_2(\mathbb{Z}) \cong D_4 *_{D_2} D_6.$$

VIII. CONCLUSION

We have shown that semidirect products preserve free products with amalgamation through both algebraic and categorical approaches. This provides a powerful structural tool for analyzing group extensions and actions on trees.

Our results:
Unify semidirect and amalgamated constructions,
Extend bass–serre theory,
Give alternative proofs of classical decompositions.

Future work may explore:
Higher-dimensional analogues,
Applications in geometric group theory,
Computational group decompositions.

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