IJARSCT
ISSN (Online) 2581-9429
International Journal of Advanced Research in Science, Communication and Technology (IJARSCT)
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
Impact Factor: 4.819
Volume 7, Issue 1, July 2020

Inverse Approach of Cylindrical Surface Heat Source and Related Thermal Stresses

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Abstract*: In this problem***,** *heat transfer analysis in a circular plate due to axisymmetric ramp-type heating under transient temperature conditions have been discussed. The circular plate is subjected to axisymmetric ramp-type heating on upper and lower surface whereas the convection due to dissipation takes place through circular boundary surface. The Laplace transform and corresponding inversion integral methods are used to remove time variable whereas Hankel transforms are used to remove space variable r. The analytical solution of boundary value problem for temperature change within circular plate have been computed numerically and illustrated graphically.*

Keywords: *axisymmetric ramp*.

I. INTRODUCTION

Nowacki [1] has determined steady-state thermal stresses in a thick circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and circular edge. Wankhede [4] has determined the quasi-static thermal stresses in circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature. However, there aren't many investigations on transient state. Roy Choudhuri [3] has succeeded in determining the quasi-static thermal stresses in a circular plate subjected to transient temperature along the circumference of circular upper face with lower face at zero temperature and the fixed circular edge thermally insulated. Warsha K. Dange, has studied [5] two-dimensional transient thermoelastic problem for a hollow cylinder with internal heat generation and [6] two-dimensional transient thermoelastic problem of a annular disc due to radiation . In this problem**,** heat transfer analysis in a circular plate due to axisymmetric ramp-type heating under transient temperature conditions have been discussed. The circular plate is subjected to axisymmetric ramp-type heating on upper and lower surface whereas the convection due to dissipation takes place through circular boundary surface. The Laplace transform and corresponding inversion integral methods are used to remove time variable whereas Hankel transforms are used to remove space variable *r*.

BOUNDARY VALUE PROBLEM OF HEAT CONDUCTION

Ramp-type heating on the upper surface is given in the form of Heaviside unit step function

$$f(r) = T_0 \cdot g(r) \cdot [H(r) - H(r - r_0)] \qquad (1)$$

where $H(r)$ is the Heaviside unit step function and $g(r)$ is arbitrary function of space variable r.

Consider a thick circular plate of radius a and thickness h defined by $0 \le r \le a$, $-\frac{h}{2} \le z \le \frac{h}{2}$. Let the plate be subjected to a transient axisymmetric temperature field on the radial and axial directions of the cylindrical coordinate system. Initially the plate is kept at zero temperature. The arbitrary heat flux $\frac{Qf(r)}{\lambda}$ is prescribed over the upper surface $(z = \frac{h}{2})$ and the lower surface $(z = -\frac{h}{2})$. The convection due to dissipation takes place through fixed circular edge $(r = a)$. Under these more realistic prescribed conditions, the transient thermal stresses are required to be determined.

The temperature of the plate at time t satisfies the heat conduction equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2)$$

with the conditions

$$\frac{\partial T}{\partial z} = \pm\frac{Qf(r)}{\lambda} \qquad \text{for } z = \pm\frac{h}{2},\ 0 \le r \le a \text{, for all time } t \quad (3)$$

$$\frac{\partial T}{\partial r} + hT = 0 \qquad \text{at } r = a,\ -\frac{h}{2} \le z \le \frac{h}{2} \qquad (4)$$

and

$$T = 0 \quad \text{at} \quad t = 0 \qquad (5)$$

where α is the thermal diffusivity of the material of the plate.

One obtains the expressions for displacement and thermal stresses for plane stress i.e. solid thin circular plate with radius *a* are

$$u = (1+\nu)a_t\left[\frac{1}{r}\int_a^r \tau r dr + \left(\frac{1-\nu}{1+\nu}\frac{r}{a^2}\right)\left(\int_0^a \tau r dr\right)\right] \qquad (6)$$

$$\sigma_{rr} = a_t E\left[\frac{-1}{r^2}\int_a^r \tau r dr + \left(\frac{1}{a^2}\int_0^a \tau r dr\right)\right] \qquad (7)$$

$$\sigma_{\theta\theta} = a_t E\left[\frac{1}{r^2}\int_a^r \tau r dr + \left(\frac{1}{a^2}\int_0^a \tau r dr - \tau\right)\right] \qquad (8)$$

$$\sigma_{zz} = \sigma_{r\theta} = 0 \quad (9)$$

Where

$$\sigma_{rr} = 0 \quad \text{at} \quad r = a$$

Equations (1) to (9) constitute mathematical formulation for heat conduction.

THE SOLUTION OF HEAT CONDUCTION PROBLEM

Taking Laplace transform of equations (2) to (4) w.r.t. t and using equation (5), one obtains

$$\frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}}{\partial r} + \frac{\partial^2 \overline{T}}{\partial z^2} = \frac{P}{\alpha}\overline{T} \qquad (10)$$

with boundary conditions

$$\frac{\partial \overline{T}}{\partial z} = \pm\frac{Qf(r)}{\lambda P} \quad \text{at } z = \pm\frac{h}{2} \qquad (11)$$

$$\frac{\partial \overline{T}}{\partial r} + h\overline{T} = 0 \quad \text{at } r = a \qquad (12)$$

where P is Laplace transform parameter.

Now applying method of separation of variables to solve equation (3.2.1), one obtain

$$\overline{T} = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \cosh[(\sqrt{\alpha_n^2 + q^2})Z] \qquad (13)$$

where $q^2 = \dfrac{P}{\alpha}$ and

α_n is n^{th} positive root of transcendental equation $J_1(\alpha \cdot a) + hJ_0(\alpha \cdot a) = 0$

Now constant A_m can be obtained by using equations (11) as

$$A_n = \frac{2Q\,\overline{f}(\alpha_n)}{a^2 \lambda P \sqrt{\alpha_n^2 + q^2}\, J_0^2(\alpha_n a) \sinh[(\sqrt{\alpha_n^2 + q^2})\frac{h}{2}]}$$

where $\overline{f}(\alpha_n) = \int_0^a r J_0(\alpha_n r) f(r)\, dr$

Using value of constant A_n in equation (13), one obtains

$$\overline{T} = \sum_{n=1}^{\infty} \left[\frac{2Q\,\overline{f}(\alpha_n)\, J_0(\alpha_n r) \cosh[(\sqrt{\alpha_n^2 + q^2})z]}{a^2 \lambda P \sqrt{\alpha_n^2 + q^2}\, J_0^2(\alpha_n a) \sinh[(\sqrt{\alpha_n^2 + q^2})\frac{h}{2}]} \right] \qquad (14)$$

Now consider inversion integral

$$I = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[\frac{\cosh\left[\left(\sqrt{\alpha_n^2 + \frac{p}{\alpha}}\right) z\right]}{p\left(\sqrt{\alpha_n^2 + \frac{p}{\alpha}}\right) \sinh\left[\left(\sqrt{\alpha_n^2 + \frac{p}{\alpha}}\right) h\right]} \right] dp$$

where C is greater than real part of all the singularities of the integrand. It is evident that there is no branch point of the integrand. Hence using Bromwich contour the integral is equal to the sum of residue at $p = 0$, $p = -\alpha_m^2 \alpha$ and of the zeros of $\sinh\left[\left(\sqrt{\alpha_n^2 + \frac{p}{\alpha}}\right)\frac{h}{2}\right] = 0$. It can also be shown that the integral round the circular arc is tends to zero.

$$\text{Residue at pole } (\mathrm{p}=0) = \lim_{p \to 0} \left[\frac{e^{pt}(p-0)\cosh\left[\left(\sqrt{\alpha_n^2 + \frac{p}{\alpha}}\right) z\right]}{p\left(\sqrt{\alpha_n^2 + \frac{p}{\alpha}}\right) \sinh\left[\left(\sqrt{\alpha_n^2 + \frac{p}{\alpha}}\right)\frac{h}{2}\right]} \right]$$

$$= \frac{\cosh(\alpha_n z)}{\alpha_n \sinh(\alpha_n \frac{h}{2})}$$

$$\text{Residue at pole } \left(\mathrm{p}=-\alpha_n^2\alpha\right) = \lim_{p\to-\alpha_n^2\alpha}\left[\frac{e^{pt}\left(p+\alpha_n^2\alpha\right)\cosh\left[\left(\sqrt{\alpha_n^2+\frac{p}{\alpha}}\right)z\right]}{\left(\sqrt{\alpha_n^2+\frac{p}{\alpha}}\right)\sinh\left[\left(\sqrt{\alpha_n^2+\frac{p}{\alpha}}\right)\frac{h}{2}\right]}\right]$$

$$=\frac{-2e^{-\alpha_n^2\alpha t}}{\alpha_n^2 h}$$

$$\text{Residue at pole } \left(\mathrm{p}=-\alpha\left(\frac{2m\pi}{h}+\alpha_n^2\right)=n(\text{say})\right)$$

$$=\lim_{p\to-n}\left[\frac{e^{pt}\left(p-n\right)\cosh\left[\left(\sqrt{\alpha_n^2+\frac{p}{\alpha}}\right)z\right]}{p\left(\sqrt{\alpha_n^2+\frac{p}{\alpha}}\right)\sinh\left[\left(\sqrt{\alpha_n^2+\frac{p}{\alpha}}\right)\frac{h}{2}\right]}\right]$$

$$=\lim_{p\to-n}\left[\frac{\left(p-n\right)}{\left(\sqrt{\alpha_n^2+\frac{p}{\alpha}}\right)\sinh\left[\left(\sqrt{\alpha_n^2+\frac{p}{\alpha}}\right)\frac{h}{2}\right]}\right]\left[\frac{e^{pt}\cosh\left[\left(\sqrt{\alpha_n^2+\frac{p}{\alpha}}\right)z\right]}{p}\right]$$

Applying L-Hospital rule for first bracket which is in $\frac{0}{0}$ form, one obtain

$$=4h\sum_{n=1}^{\infty}\left(\frac{(-1)^{m+1}\cos\left(\frac{2m\pi z}{h}\right)e^{-\alpha\left[(4m^2\pi^2+\alpha_n^2h^2)\frac{t}{h^2}\right]}}{(4m^2\pi^2+\alpha_n^2h^2)}\right)$$

$L^{-1}\left(\overline{T}\right)=$ Sum of all residues

$$L^{-1}\left(\overline{T}\right)=\frac{\cosh(\alpha_n z)}{\alpha_n\sinh(\alpha_n\frac{h}{2})}-\frac{2e^{-\alpha_n^2\alpha t}}{\alpha_n^2 h}+4h\sum_{n=1}^{\infty}\left(\frac{(-1)^{m+1}\cos\left(\frac{2m\pi z}{h}\right)e^{-\alpha\left[(4m^2\pi^2+\alpha_n^2h^2)\frac{t}{h^2}\right]}}{(4m^2\pi^2+\alpha_n^2h^2)}\right)$$

Using inverse Laplace transform of equation (14), one obtain the expression for temperature,

$$T=\left(\frac{2Q}{a^2\lambda}\right)\sum_{n=1}^{\infty}\left\{\frac{\bar{f}(\alpha_n)J_0(\alpha_n r)}{J_0^2(\alpha_n a)}\left[\frac{\cosh(\alpha_n z)}{\alpha_n\sinh(\frac{\alpha_n h}{2})}-\frac{2e^{-\alpha_n^2\alpha t}}{\alpha_n^2 h}+4h\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\cos\left(\frac{2m\pi z}{h}\right)e^{-\alpha\left[(4m^2\pi^2+\alpha_n^2h^2)\frac{t}{h^2}\right]}}{(4m^2\pi^2+\alpha_n^2h^2)}\right]\right\} \tag{15}$$

Since initial temperature $T_i=0$,

$$\tau=T \tag{16}$$

Equations (15) and (16) gives temperature change within circular cylinder due to ramp-type axisymmetric heating.

DISPLACEMENT AND THERMAL STRESSES FOR THIN SOLID PLATE

Using equations (15) and (16) in the equations (6) to (9), one obtains the expression for displacement and thermal stresses in thin circular solid as

$$u=\left(\frac{2Q(1+\nu)a_t}{a^3\lambda}\right)\sum_{n=1}^{\infty}\left\{\left(\frac{\bar{f}(\alpha_n)}{\alpha_n J_0^2(\alpha_n a)}\right)\left[\frac{a(1+\nu)J_1(\alpha_n r)+(1-\nu)rJ_1(\alpha_n a)}{(1+\nu)}\right]\times\left[\frac{\cosh(\alpha_n z)}{\alpha_n\sinh(\frac{\alpha_n h}{2})}-\frac{2e^{-\alpha_n^2\alpha t}}{\alpha_n^2 h}+4h\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\cos\left(\frac{2m\pi z}{h}\right)e^{-\alpha\left[(4m^2\pi^2+\alpha_n^2h^2)\frac{t}{h^2}\right]}}{(4m^2\pi^2+\alpha_n^2h^2)}\right]\right\} \tag{17}$$

$$\sigma_{rr}=\left(\frac{2QEa_t}{a^3\lambda}\right)\sum_{n=1}^{\infty}\left\{\left(\frac{\bar{f}(\alpha_n)}{\alpha_n J_0^2(\alpha_n a)}\right)\left[\frac{rJ_1(\alpha_n a)-aJ_1(\alpha_n r)}{r}\right]\times\left[\frac{\cosh(\alpha_n z)}{\alpha_n\sinh(\frac{\alpha_n h}{2})}-\frac{2e^{-\alpha_n^2\alpha t}}{\alpha_n^2 h}+4h\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\cos\left(\frac{2m\pi z}{h}\right)e^{-\alpha\left[(4m^2\pi^2+\alpha_n^2h^2)\frac{t}{h^2}\right]}}{(4m^2\pi^2+\alpha_n^2h^2)}\right]\right\} \tag{18}$$

$$\sigma_{\theta\theta}=\left(\frac{2QEa_t}{a^3\lambda}\right)\sum_{n=1}^{\infty}\left\{\left(\frac{\bar{f}(\alpha_n)}{\alpha_n J_0^2(\alpha_n a)}\right)\left[\frac{aJ_1(\alpha_n r)+rJ_1(\alpha_n a)-\alpha_n raJ_0(\alpha_n r)}{r}\right]\times\left[\frac{\cosh(\alpha_n z)}{\alpha_n\sinh(\frac{\alpha_n h}{2})}-\frac{2e^{-\alpha_n^2\alpha t}}{\alpha_n^2 h}+4h\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\cos\left(\frac{2m\pi z}{h}\right)e^{-\alpha\left[(4m^2\pi^2+\alpha_n^2h^2)\frac{t}{h^2}\right]}}{(4m^2\pi^2+\alpha_n^2h^2)}\right]\right\} \tag{19}$$

NUMERICAL CALCULATIONS

SPECIAL CASE

Setting

$$g(r) = 1 \quad 0 < r < r_0$$
$$= 0 \quad r > r_0$$

where $\quad 0 < r_0 < a$

$$\overline{f}(\alpha_n) = \int_0^a f(r) \cdot rJ_0(\alpha_n r)\, dr$$

$$\overline{f}(\alpha_n) = \int_0^{r_0} T_0 \cdot rJ_0(\alpha_n r)\, dr$$

$$\overline{f}(\alpha_n) = T_0\left(\frac{r_0 J_0(\alpha_n r_0)}{\alpha_n}\right)$$

DIMENSIONS

Radius of plate $a = 1m$,

For thin circular plate $h = 0.1m$,

$r_0 = 0.5m$.

ROOTS OF TRANSCENDENTAL EQUATION

Let α_i 's are the common roots of transcendental equations $J_1(\alpha \cdot a) + hJ_0(\alpha \cdot a) = 0$ given by

$$\alpha_1 = 2.6234, \quad \alpha_2 = 5.9301, \quad \alpha_3 = 8.8762, \quad \alpha_4 = 12.04, \quad \alpha_5 = 14.9678,$$

MATERIAL PROPERTIES

The numerical calculation has been carried out for a copper (Pure) circular plate with the material properties as,

Density $\rho = 8954 \ (kg/m^3)$,

Specific heat $c_p = 383 \ (J/kgK)$,

Thermal diffusivity $\alpha = 112.34 \times 10^{-6} \ (m^2 s^{-1})$,

Thermal conductivity $k=386 \ (W/mk)$,

Poisson ratio $\nu = 0.35$,

Coefficient of linear thermal expansion $a_t = 16.5 \times 10^{-6} \ (1/K)$,

Lamé constant $\mu = 26.67$,

Young's modulus $E = 130GPa.$

The numerical calculation has been carried out by setting for convenience,

$$\chi = \frac{2Q}{a^2\lambda}, \gamma = \left(\frac{16(1+\nu)\alpha}{a^2\lambda}\right), \psi = \left(\frac{8\alpha E}{a^2\lambda}\right) \text{ and } \xi = \left(\frac{8\alpha E}{a^2\lambda(1-\nu)}\right).$$

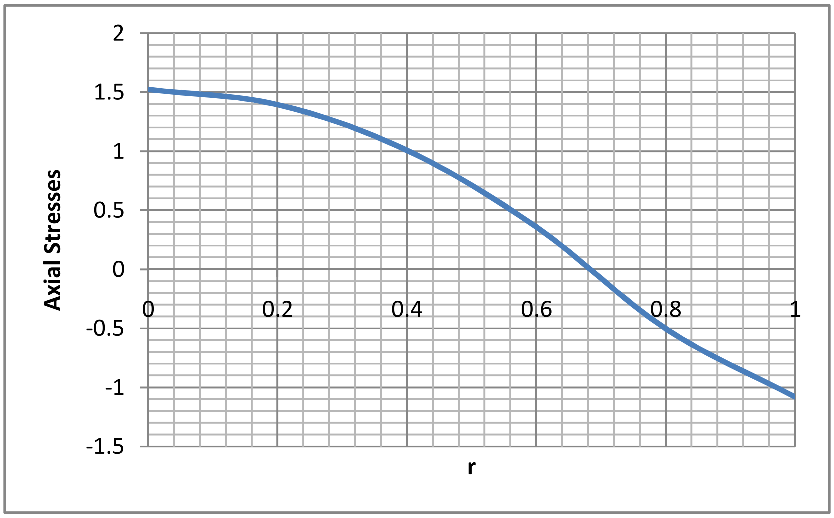

Figure 1 : Axial stress function $\frac{\sigma_{zz}(r,z,t)}{\xi}$ along radial direction.

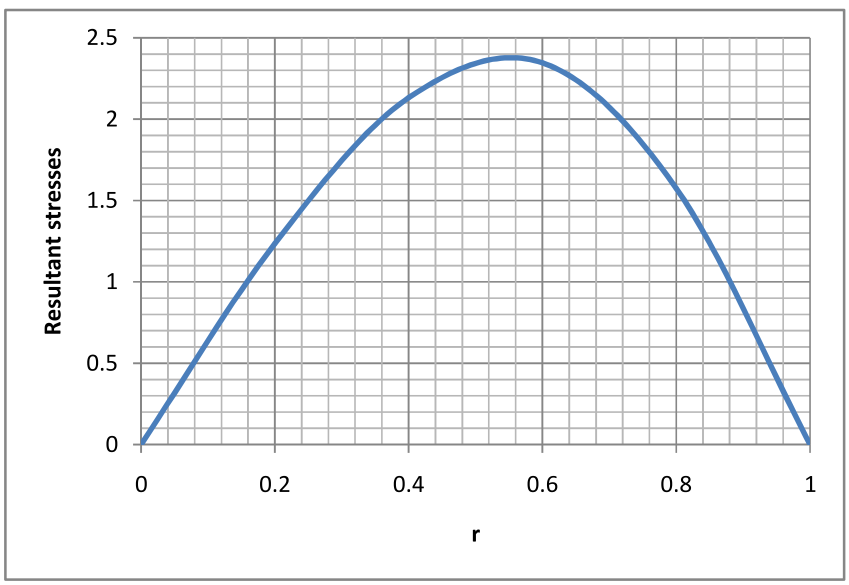

Figure 2: Resultant stress function $\frac{\sigma_{rz}(r,z,t)}{\xi}$ along radial direction.

II. CONCLUSION

In this problem a circular plate which is subjected to a transient axisymmetric temperature field on the radial and axial directions of the cylindrical coordinate system is considered and determined the expressions for temperature, displacements and stress functions, due to axisymmetric ramp-type heating . As a special case mathematical model is constructed for copper (pure) and performed numerical calculations. The thermoelastic behavior is examined such as temperature change, displacements and stresses for copper (pure) with physical properties as above.

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