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A Trek on Non-homogeneous Ternary Cubic Equation $3x^2 + 5y^2 = 128z^3$

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Abstract: *This paper concerns with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous ternary cubic equation* $3x^2 + 5y^2 = 128z^3$. *Different sets of integer solutions are illustrated*

Keywords: Non-homogeneous cubic, Ternary cubic, Integer solutions

I. INTRODUCTION

The study of integer solutions to polynomial Diophantine equations is as old as mathematics itself. It is a treasure house in which the search for many hidden connections is a treasure hunt. It has a rich history and is an active area of current research. While attempting to understand the theory of elliptic curves, an area of attraction to mathematicians since antiquity, the polynomial Diophantine equations of third degree with many variables [1-31] are noticed. These problems motivated us for searching integer solutions to a special class of ternary cubic polynomial Diophantine equation $3x^2 + 5y^2 = 128z^3$.

METHOD OF ANALYSIS:

The non-homogeneous ternary cubic equation under consideration is

$$3x^2 + 5y^2 = 128z^3 \tag{1}$$

To start with , by scrutiny , (1) is satisfied by the following choices of solutions:

Choice 1 : $x = s^{3t}, y = 5s^{3t}, z = s^{2t}$

Choice 2 : $x = 6s^{3t}, y = 2s^{3t}, z = s^{2t}$

However , there are other patterns of integer solutions to (1) that are illustrated as below:

Introduction of the linear transformations

$$x = 4(X - 5T), y = 4(X + 3T), X \neq 5T, -3T \tag{2}$$

in (1) leads to

$$X^2 + 15T^2 = z^3 \tag{3}$$

We solve (3) through different ways and using (2), obtain different sets of solutions to (1).

Way 1:

Let

$$z = a^2 + 15b^2 \tag{4}$$

Substituting (4) in (3) and employing the method of factorization, consider

$$X + i\sqrt{15}\,T = \left(a + i\sqrt{15}\,b\right)^3 = [f(a,b) + i\sqrt{15}\,g(a,b)] \tag{5}$$

where

$$\begin{aligned} f(a,b) &= a^3 - 45ab^2 \\ g(a,b) &= 3a^2b - 15b^3 \end{aligned} \tag{6}$$

On equating the real and imaginary parts in (5), one has

$$X = f(a,b)$$

$$T = g(a,b)$$

In view of (2), the integer solutions to (1) are given by

$$x = 4[f(a,b) - 5g(a,b)]$$

$$y = 4[f(a,b) + 3g(a,b)]$$

jointly with (4).

Way 2:

Rewrite (3) as

$$X^2 + 15T^2 = z^3 * 1 \tag{7}$$

Consider 1 on the R.H.S. of (8) as

$$1 = \frac{(1 + i\sqrt{15})(1 - i\sqrt{15})}{16} \tag{8}$$

Assume

$$z = 4(a^2 + 15b^2) \tag{9}$$

Substitute (8) & (9) in (7). Employing the method of factorization and equating the positive factors, one has

$$\begin{aligned} X + i\sqrt{15}\,T &= \frac{(1 + i\sqrt{15})}{4} 2^3[f(a,b) + i\sqrt{15}\,g(a,b)] \\ &= 2(1 + i\sqrt{15})[f(a,b) + i\sqrt{15}\,g(a,b)] \end{aligned}$$

from which one has

$$X = 2[f(a,b) - 15g(a,b)]$$

$$T = 2[f(a,b) + g(a,b)]$$

In view of (2), the integer solutions to (1) are given by

$$\begin{aligned} x &= 4[-8f(a,b) - 40g(a,b)] \\ y &= 4[8f(a,b) - 24g(a,b)] \end{aligned} \tag{10}$$

jointly with (9).

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Note 1:

The integer 1 on the R.H.S. of (7) is also expressed as

$$1=\frac{(15r^2-s^2+i\sqrt{15}2rs)(15r^2-s^2-i\sqrt{15}2rs)}{(15r^2+s^2)^2},$$

$$1=\frac{(10s^2-10s+1+i\sqrt{15}(2s-1))(10s^2-10s+1-i\sqrt{15}(2s-1))}{(10s^2-10s+4)^2},$$

$$1=\frac{(6s^2-6s-1+i\sqrt{15}(2s-1))(6s^2-6s-1-i\sqrt{15}(2s-1))}{(6s^2-6s+4)^2}$$

Repeating the above process, different sets of integer solutions to (1) are obtained.

Way 3:

Note that (3) is satisfied by

$$X=m\left(m^2+15n^2\right)$$
$$T=n\left(m^2+15n^2\right)$$
$$z=\left(m^2+15n^2\right)$$

From (2), we have

$$x=4\left(m-5n\right)(m^2+15n^2)$$
$$y=4\left(m+3n\right)(m^2+15n^2)$$

Thus,the above values of x, y, z satisfy (1).

Way 4

Taking

$$y=8kz \tag{11}$$

in (1), we have

$$3x^2=64z^2(2z-5k^2) \tag{12}$$

After performing some algebra, the sequence of values of z, x satisfying (12) are given by

$$\begin{aligned} z&=(6s^2-6s+4)\,k^2\\ x&=8(2s-1)(6s^2-6s+4)k^3 \end{aligned} \tag{13}$$

From (11), we get

$$y=8\,(6s^2-6s+4)k^3 \tag{14}$$

Thus, (14) & (13) satisfy (1).

Way 5

Taking

$$x=6kz \tag{15}$$

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in (1),we have

$$5\,y^2 = 4z^2(32z - 27k^2) \tag{16}$$

After performing some algebra, the values of z, y satisfying (16) are given by

$$\left.\begin{aligned} z &= (40s^2 - 75s + 36)k^2 \\ y &= 2(40s^2 - 75s + 36)(16s - 15)k^3 \end{aligned}\right\} \tag{17}$$

From (15), we get

$$x = 6\,k^3\,(40s^2 - 75s + 36) \tag{18}$$

Thus,(18) & (17) satisfy (1).

Way 6

The insertion of the transformation

$$x = k\,y \tag{19}$$

in (1) leads to

$$(3k^2 + 5)\,y^2 = 128\,z^3 \tag{20}$$

which is satisfied by

$$y = 32(3k^2 + 5)s^{3t}, z = 2(3k^2 + 5)s^{2t} \tag{21}$$

From (19), one has

$$x = 32k(3k^2 + 5)s^{3t} \tag{22}$$

Thus , (21) and (22) represent the integer solutions to (1) .

Way 7

The insertion of the transformation

$$y = k\,x \tag{23}$$

in (1) leads to

$$(3 + 5k^2)\,x^2 = 128z^3 \tag{24}$$

which is satisfied by

$$x = 32(3 + 5k^2)s^{3t}, z = 2(3 + 5k^2)s^{2t} \tag{25}$$

From (23), one has

$$y = 32k(3 + 5k^2)s^{3t} \tag{26}$$

Thus , (25) and (26) represent the integer solutions to (1) .

Remark

It is worth to mention that ,in addition to (2), one may also consider the linear transformations as

$$x = 4(X + 5T), y = 4(X - 3T), X \neq -5T, 3T$$

Following the above procedures, more sets of integer solutions to (1) are obtained.

III. CONCLUSION

An attempt has been made to obtain non-zero distinct integer solutions to the non-homogeneous cubic Diophantine equation with three unknowns given by $3x^2 + 5y^2 = 128z^3$. One may search for other sets of integer solutions to the considered equation as well as other choices of the third degree Diophantine equations with multi-variables.

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