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Painlevé Analysis to Test the Integrability of Nonlinear ODEs

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Abstract:*Many important physical equations aroused in engineering and applications are represented as Nonlinear ODE or System of nonlinear ODEs or can be simplified to the nonlinear system of ODEs. So, it has become an important task to check integrability of such nonlinear ODEs or system of ODEs. In literature it is discovered that the integrability and singularity structure of a differential equation's solution are closely connected. Multiple values around movable singularities of solution shows that the corresponding differential equation is not integrable. An ordinary differential equation is said to have the Painlev´e property (or P-type) if the only movable singularities of all its solutions are poles that is there are no movable branch points or movable essential singularities. Painlevé algorithm is the process of determining the necessary conditions for the Painleve property. This Painlevé test uses the singularity structure to demonstrate integrability of many nonlinear ODEs and also helps to find the specific parameter valus for integrability. In this study Painlevé algorithm is described and applied to identify singularity structure of solution and to demonstrate integrability of the nonlinear ordinary differential equations.*

Keywords: Painlevé Algorithm, movable singularity, nonlinear ordinary differential equations (ODEs).

I. INTRODUCTION

For adequate initial data, we believed that if the differential equation could be solved using an associated single-valued linear problem, it was known to be integrable.

But, when it comes to this concept, it is never clear what does it means (Desale & Patil, 2019). However, in literature it is observed that integrability and singularity structure of solution of differential equations are strongly associated with each other. This association was first found in 1889 in the two innovative papers (1889a) and (1889b) of Kowalevsky. Her work has shown that the system of ODEs controlling the motion of a spinning top under the effect of gravity is completely integrable by pointing out the possibilities for parameters for which the equations of motion of spinning top are independent of movable critical points. The method used by Kowalevsky has laid foundation for further development of theory of integrability of ODEs using singularity pattern of its solution. We can find these in the (Ablowitz et al., 1980).

Years later Paul Painlevé (1900, 1902) Painlevé initiated the classification of first and second order ODE's whose solutions do not possess movable branch points or movable essential singularities in the complex domain.

A differential equation possessing this Painlevé property is called Painlevé type. The results by Painlevé, along with modifications by Gambier (1910) and Fuchs (1907) lead to fifty canonical equations of the form $w'' = F(z,w,w')$ unique up to Mobius transformations with this property. It is observed that, forty-four had general solutions in the form of elliptic functions, classical special functions (specified by linear equations), or could be reduced to one of the six nonlinear ordinary differential that remain. These remaining six nonlinear ordinary differential equations are known as Painlevé equations, and these equations are represented by PI–VI (Kruskal et al., 2013) .

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PI $\quad w'' = 6\,w^2 + z,$

PII $\quad w'' = 2\,w^3 + zw + \alpha,$

PIII $\quad w'' = \frac{1}{w}w'^2 - \frac{1}{z}w' + \frac{1}{z}(\alpha w^2 + \beta) + \gamma\, w^3 + \frac{\delta}{w},$

PIV $\quad w'' = \frac{1}{2w}w'^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w},$

PV $\quad w'' = \left\{\frac{1}{2w} + \frac{1}{w-1}\right\}w'^2 - \frac{1}{z}w' + \frac{(w-1)^2}{z^2}\left(\alpha + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1},$

PVI $\quad w'' = \frac{1}{2}\left\{\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right\}w'^2 - \left\{\frac{1}{z} + \frac{1}{w-1} + \frac{1}{w-z}\right\}w' +$

$$\frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\}$$

The Painlevé analysis is based on the local singularity structure and contributes significantly in examining the singularity structure of solution to nonlinear ordinary differential equations in order to determine their integrability. Branching about movable singularities of solution to nonlinear ordinary differential equation indicates nonintegrability. Such branching is omitted in Painlevé property and hence identified as very effective technique to test integrability (Lu, 2005). The procedure followed to define the necessary conditions for the Painlevé property is known as the Painlevé test. This three step algorithm to Painlevé test in the light of ARS (Ablowitz et al., 1980). conjucture is also known as Painlevé algorithm. This analysis seeks a meromorphic series solution locally around singularities (Desale & Patil, 2010; Desale &Srinivasan, 2008). The convergence of Laurent series solution obtained by Painlevé test is guaranteed by Kichenassammy and Litteman (Ziglin, 1982). To work out whether a nonlinear ODE or system of ODEs encompasses movable algebraic or movable logarithmic branch points, the ARS algorithm was first designed. Even if an ODE does not have branch points that are movable, it can nonetheless accept essential singularities of movable type.ARS approach just offers the prerequisites for an ODE to possess Painleve property because it fails to detect singularities which are essential. In other words, it does not offer sufficient conditions for an ODE to possess Painleve property only essential prerequisites are provided by this approach. In literature this powerful technique is effectively applied to find the integrable cases of nonlinear ODE's and System of nonlinear ODE's.
In this study Painlevé algorithm is described and used to evaluate the integrability of nonlinear ODE's.

II. BASIC DEFINITIONS

Definition 2.1. An essential type singularity or a point in the neighbourhood of which the solution is multiple valued is known as the critical point of an ODE solution.

Definition 2.2. (Kruskal et al., 2013) An ODE or system of ODE's is said to have the Painlevé property if all solutions are single valued in the neighborhood of all the movable singularities of its solution and in the vicinity of the complex domain.

Definition 2.3. (Goldstein, 1998) The type of singularities of solutions to ODE's whose position is the independent of the initial conditions (or boundary conditions) are known as fixed singularities.

Definition 2.4.(Goldstein, 1998) The type of singularities of solutions to ODE's whose position is dependent on the initial conditions (or boundary conditions) are known as movable singularities.

III. THE PAINLEVÉ ALGORITHM

The process of determining the necessary conditions for the Painlevé property is known as the Painlevé test. In literature Painlevé algorithm to Painlevé test is used to determine complete integrability of nonlinear ODEs or system of ODEs. The algorithm determines Laurent expansion

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$w(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^{n+p}$ (1)

of a solution about z_0, an arbitrary position of the singularity which can be shifted to the origin by assuming the translation $Z = z - z_0$. Here p represents leading exponent that required to be determined.

To make the presentation of Painlevé algorithm much simpler, assume that (Ablowitz et al., 1980)

(i) The nth order system of ODE's has the form

$\frac{d}{dz} w_i = F_i(z; w_1, w_2, \dots, w_n), \quad i = 1,2, \dots, n$ (2)

Where each F_i is rational in $w_1, w_2, \dots, w_n$ and analytic in z.

A particular case of this is the ODE of order n.

$\frac{d^n w}{dz^n} = F\left(z; w, w', \dots, \frac{d^{n-1} w}{dz^{n-1}}\right)$ (3)

Where F is rational in $w', w'', \dots, \frac{d^{n-1} w}{dz^{n-1}}$ and analytic in z.

(ii)The order of leading exponent and coefficient of the leading order of the function in a neighborhood of the movable singularity is algebraic that is

$w_i \sim \alpha_i (z - z_0)^{p_i}$ as $z \to z_0$ where $Re(p) < 0, \alpha_i \neq 0$ (4)

The Ablowitz et al (1980)approach serves as the foundation for the following three-step algorithm to perform the Painlevé test in light of the ARS conjecture. This analysis seeks a meromorphic series solution locally around singularities (Desale & Patil, 2010; Desale &Srinivasan, 2008).

STEP 1: Leading order analysis

To find movable singularities, equation (1) should begin with a term that explodes at z_0.

For this, substituting

$w \sim \alpha \ (z - z_0)^p$ as $z \to z_0$ (5)

where $Re(p) < 0, \alpha \neq 0$into equation (3) points that for particular values of p, two or more terms in the equation can balance each other (depending on the value of α) and the remaining terms become negligible as $z \to z_0$. The terms that balance in this manner are referred to as the leading term. The need that these terms must balance typically provides a means of determining the value of leading coefficient α(Ablowitz et al., 1980).

Example 1:

$w''' + 2ww'' - w^3 + \mu w^2 + \lambda w = 0$ (6)

Considering simplified equation by preserving only leading terms and substituting (5), the two possibilities are:

$p = -1$ and $\alpha = \frac{3}{2}$ with the leading terms w''' and $2ww''$

$p = -2$ and $\alpha = 12$ with the leading terms $2ww''$ and $-w^3$

Example 2:

$w'' = 2w(w')^2/(w^2 - 1)$ (7)

substituting (5), the only possible option is $p = -1$, here α remains completely unrestricted.

If each of the probable values of p such that each p are integers and $Re(p) < 0$,equation (5) may correspond to the initial term of the Laurent expansion (1) and it is valid in the deleted neighbourhood of singularity of movable type of ODE.

Hence the solution w(z) can be written as

$w(z) = (z - z_0)^p \sum_{i=0}^{\infty} \alpha_i (z - z_0)^i, \qquad 0 < |z - z_0| < R$ (8)

Here z_0 is the arbitrary location of a singularity.

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In leading order analysis we determined, leading power P (which has to be consistent with our assumption $Re(p) < 0$) and hence leading coefficient α_0. If (n-1) of the coefficients $\{\alpha_i\}$are also arbitrary, then we have n arbitrary constants of integration of the ODE necessary to construct the generalised solution. The values of i at which these arbitrary constants manifest are referred to as resonance.

An equation can have several values for p. Any of these values of p that are not integers indicate that w is multiple valued at z_0 and if the equation exhibits asymptotic behaviour near z_0, then it describes the dominant behavior near a movable algebraic branch point of order p.

Example 3:

$$w'' + 20\,w^4 = 0 \qquad (9)$$

substituting (5), the only possible option is $p = -\frac{2}{3}$.

This value represents branch point and indicates that the given nonlinear ODE do not pass the Painlevé test and hence does not belong to the p-type.

The successive coefficients $\{\alpha_i\}$ are determined through the recurrence relation obtained by substituting equation (8) into the ordinary differential equation. The values of coefficients number i for which the recurrence equation becomes indeterminate are called resonances(Ablowitz et al., 1980).

STEP 2: Determination of resonances

The second step of this algorithm is the determination of the resonances.The index of free coefficient in the recurrence equation are called resonances.There is also standard, direct approach for determining the location of resonances without having to calculate all the previous coefficients.

After identifying the leading-order behaviour, simplify the equation by recollecting only the leading expression from the original expression. Next, substitute the obtained expression

$$w = \alpha(z - z_0)^p + \beta(z - z_0)^{p+s} \qquad (10)$$

Where $s > 0$ into the equation.

In this case, β represents an arbitrary value coefficient. To evaluate the resonance s, all the terms from equation obtained after substitution which are linear in β are collected and require that this coefficient is zero. This condition ensures that β is free. The resulting equation is also known as the resonance equation (Ablowitz et al., 1980; Kruskal et al., 2013).

i. One root of resonance equation is always (-1), which represents the arbitrary nature of z_0.
ii. Another root in step 1 is zero if α is arbitrary.
iii. Any root with $Re(s) < 0$ should be ignored, as it contradicts the assumption that in the expansion close to z_0, $(z - z_0)^p$ is the initial term.
iv. If root s is such that s is not a real integer, then even if $Re(s) > 0$ it represents a movable branch point at z_0. Although it is still mandatory to prove that such a branch point exists, there is no necessity to carry out the algorithm any further.
v. Algebraic branch points doesn't exist if all of the resonance equation's roots are real integers with positive values for each pair (p,α) that can be found using the first step. Go on to the following step to look for logarithmic branch points.
vi. If all of the roots of the resonance equation (other from -1) are positive real integers for any pair (p,α) that can be found in step 1, then there aren't any algebraic branch points. In this case go on to the Step 3 to look for logarithmic branch points.

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vii. If all the resonances are not integers (other from -1 and perhaps 0) then the given nonlinear ODE do not pass the Painlevé test and the algorithm stops. Infact, failure of this algorithm at any one out of three steps leads to rejection and as a result the ODE is considered non-integrable.

viii. To depict an nth order ODE's general solution near to a movable pole, the resonance equation must have (n-1) distinct non-negative integer roots. If, for each pair (p, α) from step1, the resonance equation has less than (n-1) such roots then it is not possible to classify any of the local solutions as general. This indicates that equation (5) might lack a crucial component of the solution.

Example 4:

$w'' + 4\,ww' = 0$ (11)

Using Leading order analysis, $p = -1, \alpha = \frac{1}{2}$

Substituting $w = \frac{1}{2}(z - z_0)^{-1} + \beta(z - z_0)^{s-1}$ in equation (11), the resonance equation is

$s^2 - s - 2 = 0$

The roots of this resonance equation are $s = 2$ and $s = -1$

The root $s = -1$must be included in every expansion involving singularity of movable type as it represents arbitrariness of z_0 and the positive root $s = 2$ is the resonance that indicates the coefficient $\alpha_2$should be arbitrary.

STEP 3: Verification of compatibility condition.

We must look at each of the positive resonances that found in the earlier steps whether the compatibility conditions are met.

The recursion relations will be valid only in the event that the coefficient in the right side is annihilated by every zero of the left-hand side to ensure the consistency of compatibility conditions.

When these prerequisites are satisfied, the respective coefficient appears as an arbitrary constant in the expansion. However, if the compatibility condition fails at a resonant level, in the expansion logarithmic terms must be added and each instance of possible branches of leading order coefficients is analysed.

Note: An ODE may possess singularities of movable essential type even though it does not have movable branch points. Above method simply offers necessary criteria for an ODE to possess Painlevé property because essential singularities are not identified by this method.

Example 5:

$w'' = 3w^2 + z$ (12)

Laurent series expansion of the solution is given by

$w(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^{n+p}$ (13)

where z_0 is the singularity's arbitrary location in its domain.

STEP 1: Determination of leading order exponent

With the leading terms w'' and $3w^2$, consider the truncated expression from the original expression retaining only the leading terms and Substitute

$w \sim \alpha_0 \ \ (z - z_0)^p$ as $z \to z_0$ (14)

The resulting dominant equation is

$p\,(p - 1)\alpha_0(z - z_0)^{p-2} = 3\,[\alpha_0{}^2(z - z_0)^{2p} + \cdots]$ (15)

As the leading terms must balance, we have

$p - 2 = 2p$, and hence $p = -2$

Substitute $p = -2$ into (14) gives $\alpha_0 = 2$

STEP 2: Determination of resonance

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Substituting

$w(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^{n-2}$ (16)

in equation (12)

$\sum_{n=0}^{\infty}(n-2)(n-3)\alpha_n(z-z_0)^{n-4} = 3 \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\alpha_{n_1}\alpha_{n_2}(z-z_0)^{n_1+n_2-4} + (z-z_0) + z_0$ (17)

Rearranging the terms and comparing the same power of $(z - z_0)$following recurrence relation is obtained, which is valid for $n \geq 1$

$(n+1)(n-6)\alpha_n = 3 \sum_{s=1}^{n-1} \alpha_s \alpha_{n-s} + 3\,\delta_{n,5} + z_0 \delta_{n,4}$ (18)

Where δ is the Kronecker delta function, n=1,2,3,….

This recurrence relation gives the following values of the coefficients

$$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = \frac{-3}{10} z_0, \alpha_5 = \frac{-1}{2}$$

When $n = 6$ the coefficient of α_6 becomes zero, and hence equation (18) fails in determiningα_6. If the right-hand side also becomes zero, $\alpha_6$is arbitrary. However, if the right-hand side is nonzero, a contradiction arises, indicating that the series cannot be formal equation's solution (12).

In this situation, adding suitable logarithmic terms beginning at index n=6 will improve the expansion to yield a formal solution.

Throughout the expansion, logarithmic terms emerge infinite number of times and cannot be removed which is indicator of multivaluedness around movable singularities. As a result, Equation (12) fails the Painlevé test if the right-hand side of Equation (18) does not disappear at n=6.

STEP 3: The index $n = 6$of free coefficient $\alpha_6$from the above recurrence relation is the resonance for equation (12). As the right-hand side of Equation (18) is also assumed zero the compatibility condition is met.

Indeed, the recurrence relations will hold true if whenever coefficient of α_n present in the left-hand side reduces to zero, the right-hand side also reduces to zero and it give rise to the compatibility conditions to be satisfied by the coefficients.

These requirements are simply the necessary criteria for the Painlevé property to be satisfied.

If all the resonances which are nonnegative meet the compatibility requirements and every formal solutions to ODE are meromorphic around every arbitrary singularity z_0, the equation is considered to have passed the Painlevé test.

IV. RESULTS AND DISCUSSION

Detailed study of the Painlevé algorithm for the nonlinear nth order system of ODE is done and applied effectively to test the complete integrability of nonlinear ODEs.

V. CONCLUSIONS

Painlevé test plays an important role and an effective tool to test complete integrability of given Nonlinear ODE. Painlevé algorithm to Painlevé test plays an important role to find integrable cases of nonlinear ODEs and gives necessary condition for an ODE being of Painlevé type.

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