IJARSCT

International Journal of Advanced Research in Science, Communication and Technology

International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal

ISSN: 2581-9429 Volume 2, Issue 1, May 2023 Impact Factor: 7.301

Tightness and Generation of Premeasure Spaces

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Abstract: *The present paper deals with the theory of [0, 1] valued maps defined on a nonempty set X. We have concentrated over the study of additive functions, tightness and generation of premeasure spaces.*

Keywords: *premeasure spaces.*

I. INTRODUCTION

In measure theory, a basic procedure is that of extending the notion of a "measure" on a given class of sets to a larger class of sets. Kelley, Nayak and Srinivasan [**4**] proved that a nonnegative real valued function μ defined on a lattice $\mathcal{A}$ of sets is a premeasure (meaning that it extends to a countably additive measure on a δ-ring of sets containing $\mathcal{A}$) provided μ is tight and continuous at ϕ. The extension of this theorem to the case of a real valued (not necessarily nonnegative real valued) function is dealt in [**9**]. In 1981, Morales [**8**] established a quite general extension theorem for a uniform semi_group-valued tight set function λ on a lattice $\mathcal{L}$, the domain of extension being the σ-ring generated by $\mathcal{L}$. He also discussed the extension of λ on the σ-algebra of locally $\mathcal{L}$-measurable sets. The problem of generation of measures by tight functions defined on a lattice of sets has been taken up by several workers [**2, 6, 12, 13, 14**]. Adamski [**1**] proved that every nonnegative, semifinite, smooth at ϕ, tight function defined on a lattice of sets can be extended to an inner regular measure. Besides proving results on the approximation of measurable sets by members of a lattice $\mathcal{A}$, Kelley and Srinivasan [**5**] proved that every function $\mu : \mathcal{A} \to R^+$, which is tight and smooth from above at ϕ is a premeasure (here $\mathcal{A}$ is closed under countable intersections). In [**9**], a weaker condition for tightness than in [**5**] is used, aiming at its adaptation to the vector valued case. Recently we have obtained a Jordan decomposition type theorem for a weakly tight function under suitable conditions [**7**].

The present paper discusses additive functions, tightness and generation of premeasure spaces.

In Section 2, we define superadditive and subadditive functions using pointwise addition of elements in I^X. The lower envelope of a superadditive function β defined on K (K is a lattice in I^X which is closed under addition) turns out to be superadditive. But to obtain the upper envelope of a function β to be subadditive, we need the concept of addition modulo 1. Two lattices K and L in I^X are linked with different suitable conditions to obtain $\gamma = \beta_* \mid \mathcal{L}$ to be supertight and supercotight, and $\gamma = \beta^* \mid \mathcal{L}$ to be subcotight. In particular, every monotone and superadditive function (subadditive modulo 1 function respectively) defined on a lattice in I^X which is closed under addition (closed under addition modulo 1 respectively) is supertight (subcotight respectively).

Sections 3 and 4 are concerned with the study of premeasure spaces. For a given premeasure space $(X, \mathcal{K}, \beta)$ and a map $T : X \to Y$ where Y is a set, the coinduced system $(Y, \mathcal{L}_T, \gamma_T)$ is constructed; γ_T preserves the properties of β such as, monotone, modular, supertight, subcotight, smooth from below and smooth from above. Likewise, for a map $T : X \to (Y, \mathcal{L}, \gamma)$ where X is a set, the induced system $(X, \mathcal{K}_T, \beta_T)$ is obtained in Section 4. However, to retain the properties of γ, like monotone, subtight, supercotight, smooth from below and smooth from above, the condition "T

is onto" is necessary. We also observed that the coinduced system $(Y, \mathcal{L}_T, \gamma_T)$ is the largest system with respect to which T is lattice preserving and the induced system $(X, \mathcal{K}_T, \beta_T)$ is the smallest system with respect to which T becomes lattice preserving.

II. ADDITIVE FUNCTIONS AND TIGHTNESS

Notations 2.1. Throughout this chapter X is a nonempty set; $\mathcal{K}$ and $\mathcal{L}$ stand for lattices in I^X containing **0** and **1**; $\beta : \mathcal{K} \to I$ and $\gamma : \mathcal{L} \to I$ are functions satisfying $\beta(\mathbf{0}) = 0$ and $\gamma(\mathbf{0}) = 0$.

Definition 2.2. Let $\mathcal{C} \subseteq I^X$. Then $\xi : \mathcal{C} \to I$ is said to be *superadditive* (*subadditive* respectively) if, for $f_1, f_2 \in \mathcal{C}$ such that $f_1 + f_2 \in \mathcal{C}$,

$\xi(f_1 + f_2) \geq \xi(f_1) + \xi(f_2)$

($\xi(f_1 + f_2) \leq \xi(f_1) + \xi(f_2)$ respectively).

The function ξ is additive (cf. Definition 2.3 of Chapter III) iff it is both superadditive and subadditive.

Definition 2.3. The family $\mathcal{C} \subseteq I^X$ is said to be *closed under addition* if

$f, g \in \mathcal{C}, f + g \in I^X \Rightarrow f + g \in \mathcal{C}$.

Proposition 2.4. *If $\mathcal{K}$ is closed under addition and β is superadditive, then β_* is superadditive.*

Proof. Let f_1 and f_2 be in I^X such that $f_1 + f_2 \in I^X$. Let $\varepsilon > 0$. Then there exist $g_1, g_2 \in \mathcal{K}$ with $g_1 \leq f_1$ and $g_2 \leq f_2$ such that

$\beta(g_1) > \beta_*(f_1) - \varepsilon/2$

and

$\beta(g_2) > \beta_*(f_2) - \varepsilon/2$.

Hence

$\beta_*(f_1) + \beta_*(f_2) - \varepsilon < \beta(g_1) + \beta(g_2)$.

Since, for any $x \in X, 0 \leq (g_1 + g_2)(x) \leq (f_1 + f_2)(x) \leq 1$, we get $g_1 + g_2 \in I^X$ and so $g_1 + g_2 \in \mathcal{K}$. Since β is superadditive and $g_1 + g_2 \leq f_1 + f_2$,

$$\begin{aligned}\beta_*(f_1) + \beta_*(f_2) - \varepsilon &< \beta(g_1 + g_2)\\ &\leq \beta_*(f_1 + f_2).\end{aligned}$$

Therefore, we obtain

$\beta_*(f_1) + \beta_*(f_2) \leq \beta_*(f_1 + f_2)$.

Definition 2.5. For $f, g \in I^X$, define the *addition of f and g modulo* 1, written as $f \oplus g$, as an element of I^X given by

$(f \oplus g)(x) = \min\{f(x) + g(x), 1\}, x \in X$.

Definition 2.6. A subfamily $\mathcal{C}$ of I^X is said to be *closed under addition modulo* 1 (or *closed under* $\oplus$) if

$f, g \in \mathcal{C} \Rightarrow f \oplus g \in \mathcal{C}$.

A function $\xi : \mathcal{C} \to I$ is called *subadditive modulo* 1 if, for $f_1, f_2 \in \mathcal{C}$ with $f_1 \oplus f_2 \in \mathcal{C}$ the following holds:

$\xi(f_1 \oplus f_2) \leq \xi(f_1) + \xi(f_2)$.

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ISSN: 2581-9429
Volume 2, Issue 1, May 2023
Impact Factor: 7.301

Remark 2.7. If $f, g \in I^X$, $f+g \in I^X$, then $f \oplus g = f+g$. Thus if C is closed under addition modulo 1, then C is closed under addition. Also, if ξ is subadditive modulo 1, then ξ is subadditive.

Proposition 2.8. *Let K be closed under addition modulo* 1. *If β is subadditive modulo* 1, *then β^* is subadditive modulo* 1, *and hence β^* is sub- additive.*

Proof. Let $f_1, f_2 \in I^X$. Let ε > 0. Then there exist $g_1, g_2 \in \mathsf{K}$ with $f_1 \le g_1$ and $f_2 \le g_2$ such that

$\beta(g_1) < \beta^*(f_1) + \varepsilon/2$

and

$\beta(g_2) < \beta^*(f_2) + \varepsilon/2$.

It follows that

$$\beta^*(f_1) + \beta^*(f_2) + \varepsilon > \beta(g_1) + \beta(g_2). \qquad (2.8.1)$$

Since $g_1, g_2 \in \mathsf{K}$ and K is closed under ⊕, we get $g_1 \oplus g_2 \in \mathsf{K}$. Also, $f_1 \oplus f_2 \le g_1 \oplus g_2$. Hence (2.8.1) gives that

$$\begin{aligned}\beta^*(f_1) + \beta^*(f_2) + \varepsilon &> \beta(g_1 \oplus g_2)\\ &\ge \beta^*(f_1 \oplus f_2).\end{aligned}$$

Since ε is arbitrary, we obtain that β^* is subadditive modulo 1.

Definition 2.9. Let $\beta : \mathsf{K} \to I$ be a function with $\beta(\mathbf{0}) = 0$. Then β is said to be *supertight* (*supercotight* respectively) if, for $f_1, f_2 \in \mathsf{K}$ with $f_1 \le f_2$,

$$\beta(f_2) - \beta(f_1) \ge \beta_*(f_2 - f_1)$$

$$\left(\beta(f_2) - \beta(f_1) \ge \beta^*(f_2 - f_1) \text{ respectively}\right).$$

The function β is called *subtight* (*subcotight* respectively) if, for $f_1, f_2 \in \mathsf{K}$ with $f_1 \le f_2$,

$$\beta(f_2) - \beta(f_1) \le \beta_*(f_2 - f_1)$$

($\beta(f_2) - \beta(f_1) \le \beta^*(f_2 - f_1)$ respectively).

Remark 2.10. β is tight iff β is both supertight and subtight. Likewise, β is cotight iff β is both supercotight and subcotight.

Theorem 2.11. *Let K be a lattice in I^X which is closed under addition. Let L be a lattice in I^X such that $g - f \in \mathsf{L}$ whenever $g \in \mathsf{L}$, $f \in \mathsf{K}$ and $f \le g$. If β is superadditive, then $\gamma = \beta_* \mid \mathsf{L}$ is supercotight.*

Proof. Let g_1 and g_2 be in L with $g_1 \le g_2$. Let ε > 0. Then there exists $f \in \mathsf{K}$ such that $f \le g_1$ and

$$\gamma(g_1) < \beta(f) + \varepsilon. \qquad (2.11.1)$$

We also have $g_2 - g_1 \le g_2 - f$ and $g_2 - f \in \mathsf{L}$. Now let $f_1 \in \mathsf{K}$ with $f_1 \le g_2 - f$. Then $f + f_1 \le g_2$. Hence $f + f_1 \in I^X$ and so $f + f_1 \in \mathsf{K}$. Now

$$\beta(f_1 + f) \le \beta_*(g_2) = \gamma(g_2). \qquad (2.11.2)$$

Also, since β is superadditive, by (2.11.1) and (2.11.2), we get

$\gamma(g_1) + \beta(f_1) < \beta(f_1) + \beta(f) + \varepsilon$

DOI: 10.48175/IJARSCT-3601B

IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429 Volume 2, Issue 1, May 2023 Impact Factor: 7.301

$$\le \beta(f_1 + f) + \varepsilon$$
$$\le \gamma(g_2) + \varepsilon.$$

Consequently

$$\gamma(g_2) - \gamma(g_1) > \beta(f_1) - \varepsilon,$$

and so

$$\gamma(g_2) - \gamma(g_1) + \varepsilon \ge \beta_*(g_2 - f)$$
$$= \gamma(g_2 - f)$$
$$\ge \gamma^*(g_2 - g_1),$$

which gives that γ is supercotight.

Theorem 2.12. *Let K be closed under addition and let β be superadditive. Let L be a lattice in I^X which is closed under addition. Then $\gamma = \beta_* \mid L$ is supertight.*

Proof. Let $g_1, g_2 \in L$ with $g_1 \le g_2$. Since β is superadditive and K is closed under addition, by Proposition 2.4, β_* is superadditive. Since L is closed under addition, γ is also superadditive. Hence, again by Proposition 2.4, γ_* is superadditive. Now, since γ is monotone, we get

$$\gamma(g_2) = \gamma_*(g_2) = \gamma_*(g_1 + (g_2 - g_1))$$
$$\ge \gamma_*(g_1) + \gamma_*(g_2 - g_1)$$
$$= \gamma(g_1) + \gamma_*(g_2 - g_1),$$

showing that γ is supertight.

Theorem 2.13. *Every monotone and superadditive function defined on a lattice in I^X which is closed under addition is supertight.*

Proof. Follows from Theorem 2.12 and the fact that, if β is monotone then $\beta_* \mid K = \beta$.

Theorem 2.14. *Let K be closed under addition modulo* 1 *and let β be subadditive modulo* 1. *Let L be a lattice in I^X which is closed under addition modulo* 1. *Then $\gamma = \beta^* \mid L$ is subcotight.*

Proof. Let $g_1, g_2 \in L$ with $g_1 \le g_2$. Then, by Proposition 2.8, γ is subadditive modulo 1. Hence γ^* is subadditive. Now, since γ is monotone, we get

$$\gamma^*(g_2) = \gamma^*(g_1 + (g_2 - g_1))$$
$$\le \gamma^*(g_1) + \gamma^*(g_2 - g_1),$$

showing that

$$\gamma(g_2) - \gamma(g_1) \le \gamma^*(g_2 - g_1),$$

i.e. γ is subcotight.

Theorem 2.15. *Every monotone and subadditive modulo* 1 *function defined on a lattice in I^X which is closed under addition modulo* 1 *is sub-cotight.*

Proof. Follows from Theorem 2.14.

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ISSN: 2581-9429 Volume 2, Issue 1, May 2023 Impact Factor: 7.301

III. COINDUCED PREMEASURE SPACES

Definition 3.1. Let (X, K, β) be a premeasure space (i.e. X be a non empty set, K be a lattice of elements in I^X containing **0** and **1**, and $\beta : \mathsf{K} \to I$ be a function satisfying $\beta(\mathbf{0}) = 0$) and Y be a set. Let $T : X \to Y$ be a map. Define

$\mathsf{L}_T = \{g \in I^Y : T^{-1}(g) \in \mathsf{K}\}$,

where $T^{-1}(g) = g \, o \, T$.

Following identities shall be used in the sequel:

for $g_1, g_2 \in I^Y$, and for any sequence $\{f_n\}_{n=1}^{\infty}$ in I^Y,

1. $g_1 \le g_2 \Rightarrow T^{-1}(g_1) \le T^{-1}(g_2)$;
2. $T^{-1}(g_1) \le T^{-1}(g_2) \Rightarrow g_1 \le g_2$ on range T;
3. $T^{-1}(g_1 \vee g_2) = T^{-1}(g_1) \vee T^{-1}(g_2)$;
4. $T^{-1}(g_1 \wedge g_2) = T^{-1}(g_1) \wedge T^{-1}(g_2)$;
5. $g_1 \le g_2 \Rightarrow T^{-1}(g_2 - g_1) = T^{-1}(g_2) - T^{-1}(g_1)$;
6. $\{f_n\}_{n=1}^{\infty} \downarrow f, f \in I^Y \Rightarrow T^{-1}(f_n) \downarrow T^{-1}(f)$;
7. $T^{-1}(f_n) \downarrow T^{-1}(f), f \in I^Y \Rightarrow f_n \downarrow f$ on range T;
8. $\{f_n\}_{n=1}^{\infty} \uparrow f, f \in I^Y \Rightarrow T^{-1}(f_n) \uparrow T^{-1}(f)$;
9. $T^{-1}(f_n) \uparrow T^{-1}(f), f \in I^Y \Rightarrow f_n \uparrow f$ on range T;
10. $T^{-1}(\bigwedge_{n=1}^{\infty} f_n) = \bigwedge_{n=1}^{\infty} T^{-1}(f_n)$;
11. $T^{-1}\left(\bigvee_{n=1}^{\infty} f_n\right) = \bigvee_{n=1}^{\infty} T^{-1}(f_n)$.

Proposition 3.2. *The family* L_T *is a sublattice of* I^Y.

Proof. Let $f, g \in \mathsf{L}_T$. Then $T^{-1}(f)$ and $T^{-1}(g)$ belong to K. Also

$T^{-1}(f \vee g) = T^{-1}(f) \vee T^{-1}(g)$ and $T^{-1}(f \wedge g) = T^{-1}(f) \wedge T^{-1}(g)$.

Since K is a lattice, it follows that $f \vee g$ and $f \wedge g$ are in L_T, showing that L_T is a lattice.

Definition 3.3. Define a function $\gamma_T : \mathsf{L}_T \to I$ by

$\gamma_T(g) = \beta(T^{-1}(g)), g \in \mathsf{L}_T$.

We obtain the following:

Proposition 3.4. (i) $\gamma_T(\mathbf{0}) = 0$.

(ii) *If* β *is monotone, then* γ_T *is monotone.*

(iii) *If* β *is modular, then* γ_T *is modular.*

(iv) *If* β *is supertight, then* γ_T *is supertight.*

(v) *If* β *is subcotight, then* γ_T *is subcotight.*

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ISSN: 2581-9429
Volume 2, Issue 1, May 2023
Impact Factor: 7.301

(vi) *If β is smooth from above, then* γ_T *is smooth from above.*

(vii) *If β is smooth from below, then* γ_T *is smooth from below.*

Proof. (i) Since $T^{-1}(\mathbf{0}) = 0$, the assertion follows.

(ii) Let f, g in L_T with $g \le f$. Then $T^{-1}(f)$ and $T^{-1}(g)$ are in K with $T^{-1}(g) \le T^{-1}(f)$. Now since β is monotone, we get

$\gamma_T(g) = \beta(T^{-1}(g)) \le \beta(T^{-1}(f)) = \gamma_T(f)$.

(iii) Let *f* and *g* be in L_T. Then, by Proposition 3.2, $f \vee g$ and $f \wedge g$ are in L_T. Hence

$$\gamma_T(f) + \gamma_T(g) = \beta(T^{-1}(f)) + \beta(T^{-1}(g))$$

$$= \beta(T^{-1}(f) \vee T^{-1}(g)) + \beta(T^{-1}(f) \wedge T^{-1}(g))$$

$$= \beta(T^{-1}(f \vee g)) + \beta(T^{-1}(f \wedge g))$$

$$= \gamma_T(f \vee g) + \gamma_T(f \wedge g).$$

(iv) Let $g_1, g_2 \in L_T$ with $g_1 \le g_2$. Since $T^{-1}(g_2 - g_1) = T^{-1}(g_2) - T^{-1}(g_1)$, we get

$$\gamma_T(g_2) - \gamma_T(g_1) = \beta(T^{-1}(g_2)) - \beta(T^{-1}(g_1))$$

$$\ge \beta_*(T^{-1}(g_2) - T^{-1}(g_1))$$

$$= \beta_*(T^{-1}(g_2 - g_1)). \quad (3.4.1)$$

Now, let $g \in L_T$ such that $g \le g_2 - g_1$. Then $T^{-1}(g) \in K$ and $T^{-1}(g) \le T^{-1}(g_2 - g_1)$. Hence

$\gamma_T(g) = \beta(T^{-1}(g)) \le \beta_*(T^{-1}(g_2 - g_1))$.

Consequently,

$(\gamma_T)_*(g_2 - g_1) \le \beta_*(T^{-1}(g_2 - g_1))$.

Hence, by (3.4.1), we get

$\gamma_T(g_2) - \gamma_T(g_1) \ge (\gamma_T)_*(g_2 - g_1)$,

showing that γ_T is supertight.

(v) Let g_1 and g_2 be in L_T such that $g_1 \le g_2$. Then

$$\gamma_T(g_2) - \gamma_T(g_1) = \beta(T^{-1}(g_2)) - \beta(T^{-1}(g_1))$$

$$\le \beta^*(T^{-1}(g_2) - T^{-1}(g_1))$$

$$= \beta^*(T^{-1}(g_2 - g_1)). \quad (3.4.2)$$

Let $g \in L_T$ such that $g \ge g_2 - g_1$. Then $T^{-1}(g) \in K$ and $T^{-1}(g) \ge T^{-1}(g_2 - g_1)$. Hence

$\gamma_T(g) = \beta(T^{-1}(g)) \ge \beta^*(T^{-1}(g_2 - g_1))$.

Now (3.4.2) gives that

$\gamma_T(g_2) - \gamma_T(g_1) \le (\gamma_T)^*(g_2 - g_1)$,

i.e. γ_T is subcotight.

DOI: 10.48175/IJARSCT-3601B

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International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429 Volume 2, Issue 1, May 2023 Impact Factor: 7.301

(vi) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in L_T such that $\{f_n\}\downarrow f$, where $f\in\mathsf{L}_T$. Then $\{T^{-1}(f_n)\}_{n=1}^{\infty}$ is a sequence in K, $T^{-1}(f)\in\mathsf{K}$ and $T^{-1}(f_n)\downarrow T^{-1}(f)$. Since β is smooth from above, $\beta(T^{-1}(f))=\inf_n \beta(T^{-1}(f_n))$.

Hence

$$\gamma_T(f)=\beta(T^{-1}(f))$$
$$=\inf_n \beta(T^{-1}f_n)$$
$$=\inf_n \gamma_T(f_n),$$

i.e. γ_T is smooth from above.

(vii) Since $\{f_n\}\uparrow f$ implies $T^{-1}(f_n)\uparrow T^{-1}(f)$, the result follows using similar arguments as in (vi).

Definition 3.5. We call the triple $(Y,\mathsf{L}_T,\gamma_T)$ the premeasure space *coinduced* (or *generated*) by the premeasure space (X,K,β).

Proposition 3.6. *For any* $f\in I^Y$,

(i) $(\gamma_T)_*(f)\le\beta_*(T^{-1}(f))$;

(ii) $\gamma^*(f)\ge\beta^*(T^{-1}(f))$.

Proof. (i) Let $f\in I^Y$. Let $g\in\mathsf{L}_T$ such that $g\le f$. Then $T^{-1}(g)\in\mathsf{K}$ and $T^{-1}(g)\le T^{-1}(f)$. Hence

$$\gamma_T(g)=\beta(T^{-1}(g))$$
$$\le\beta_*(T^{-1}(f)).$$

It follows that

$$(\gamma_T)_*(f)\le\beta_*(T^{-1}(f)).$$

(ii) Let $f\in I^Y$. Let $g\in\mathsf{L}_T$ such that $f\le g$. Then $T^{-1}(g)\in\mathsf{K}$ and $T^{-1}(f)\le T^{-1}(g)$, which yield $\gamma_T(g)\ge\beta^*(T^{-1}(f))$. Therefore $(\gamma_T)^*(f)\ge\beta^*(T^{-1}(f))$.

IV. INDUCED PREMEASURE SPACES

Let (Y,L,γ) be a premeasure space (i.e. Y be a nonempty set, L be a sublattice of I^Y containing **0** and **1** and $\gamma:\mathsf{L}\to I$ be a function satisfying $\gamma(\mathbf{0})=0$). Let X be a set. Let $T:X\to Y$ be a map. Define

$$T^{-1}(\mathsf{L})=\{T^{-1}(g):g\in\mathsf{L}\}.$$

Let us denote $T^{-1}(\mathsf{L})$ by K_T.

Proposition 4.1. *The family* K_T *is a sublattice of* I^X.

Proof. Let $h_1,h_2\in\mathsf{K}_T$. Then there exist $f,g\in\mathsf{L}$ such that $h_1=T^{-1}(f)$ and $h_2=T^{-1}(g)$. Since L is a lattice, $f\vee g$ and $f\wedge g$ are in L and so $h_1\vee h_2=T^{-1}(f\vee g)\in\mathsf{K}_T$ and $h_1\wedge h_2=T^{-1}(f\wedge g)\in\mathsf{K}_T$. Hence K_T is a lattice.

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Definition 4.2. Define a mapping $\beta_T : K_T \to I$ by

$\beta_T(T^{-1}(g)) = \gamma(g), g \in L$.

We obtain the following:

Proposition 4.3. (i) $\beta_T(\mathbf{0}) = 0$.

(ii) *If T is onto and* γ *is monotone, then* β_T *is monotone.*

(iii) *If* γ *is modular, then* β_T *is modular.*

(iv) *If T is onto and* γ *is subtight, then* β_T *is subtight.*

(v) *If T is onto and* γ *is supercotight, then* β_T *is supercotight.*

(vi) *If T is onto and* γ *is smooth from above, then* β_T *is smooth from above.*

(vii) *If T is onto and* γ *is smooth from below, then* β_T *is smooth from below.*

Proof. (i) Since $T^{-1}(\mathbf{0}) = 0$, the assertion follows.

(ii) Let $h_1, h_2 \in K_T$ such that $h_1 \le h_2$. Then $h_1 = T^{-1}(f)$ and $h_2 = T^{-1}(g)$, for some f, g in L. Also, by 3.1(2), $f \le g$. Since γ is monotone, we get

$\beta_T(h_1) = \beta_T(T^{-1}(f)) = \gamma(f) \qquad \le \gamma(g) \qquad = \beta_T(h_2)$.

(iii) Let $h_1, h_2 \in K_T$. Then $h_1 = T^{-1}(f)$ and $h_2 = T^{-1}(g)$ for some f, g in L. Since γ is modular, we have

$$\begin{aligned}\beta_T(h_1) + \beta_T(h_2) &= \beta_T(T^{-1}(f)) + \beta_T(T^{-1}(g))\\ &= \gamma(f) + \gamma(g)\\ &= \gamma(f \vee g) + \gamma(f \wedge g)\\ &= \beta_T(T^{-1}(f \vee g)) + \beta_T(T^{-1}(f \wedge g))\\ &= \beta_T(T^{-1}(f) \vee T^{-1}(g)) + \beta_T(T^{-1}(f) \wedge T^{-1}(g))\\ &= \beta_T(h_1 \vee h_2) + \beta_T(h_1 \wedge h_2).\end{aligned}$$

(iv) Let $h_1, h_2 \in K_T$ such that $h_1 \le h_2$. Then $h_1 = T^{-1}(g_1)$ and $h_2 = T^{-1}(g_2)$ for some $g_1, g_2 \in L$. Also $g_1 \le g_2$. Now,

$$\begin{aligned}\beta_T(h_2) - \beta_T(h_1) &= \beta_T(T^{-1}(g_2)) - \beta_T(T^{-1}(g_1))\\ &= \gamma(g_2) - \gamma(g_1)\\ &\le \gamma_*(g_2 - g_1). \quad (4.3.1)\end{aligned}$$

Let $g \in L$ and $g \le g_2 - g_1$. Then $T^{-1}(g) \le T^{-1}(g_2 - g_1) = h_2 - h_1$ and hence

$\gamma(g) = \beta_T(T^{-1}(g)) \le (\beta_T)_*(h_2 - h_1)$.

It follows that

$\gamma_*(g_2 - g_1) \le (\beta_T)_*(h_2 - h_1)$.

Hence by (4.3.1), we get

$\beta_T(h_2) - \beta_T(h_1) \le (\beta_T)_*(h_2 - h_1)$,

i.e. β_T is subtight.

IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429 Volume 2, Issue 1, May 2023 Impact Factor: 7.301

(v) Let $h_1, h_2 \in K_T$ such that $h_1 \le h_2$. Then $h_1 = T^{-1}(g_1)$ and $h_2 = T^{-1}(g_2)$ for some $g_1, g_2 \in L$. Also $g_1 \le g_2$. Now

$$\beta_T(h_2) - \beta_T(h_1) = \beta_T(T^{-1}(g_2)) - \beta_T(T^{-1}(g_1))$$
$$= \gamma(g_2) - \gamma(g_1)$$
$$\ge \gamma^*(g_2 - g_1). \qquad (4.3.2)$$

Let $g \in L$ and $g \ge g_2 - g_1$. Then $T^{-1}(g) \ge T^{-1}(g_2 - g_1) = h_2 - h_1$ and hence

$\gamma(g) = \beta_T(T^{-1}(g)) \ge (\beta_T)^*(h_2 - h_1)$.

It follows that

$\gamma^*(g_2 - g_1) \ge (\beta_T)^*(h_2 - h_1)$.

Hence, by (4.3.2), we get

$\beta_T(h_2) - \beta_T(h_1) \ge (\beta_T)^*(h_2 - h_1)$,

i.e. β_T is supercotight.

(vi) Since T is onto and γ is monotone, by (ii), β_T is monotone. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $K_T, h \in K_T$ and $h_n \downarrow h$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in L such that $T^{-1}(f_n) = h_n$ for each n. Also, $h = T^{-1}(f)$ for some $f \in L$. Since f is onto, by (3.1) (7), we get $f_n \downarrow f$.

Now, since γ is smooth from above, we obtain

$$\beta_T(h) = \beta_T(T^{-1}(f)) = \gamma(f)$$
$$= \inf_n \{\gamma(f_n)\}$$
$$= \inf_n \{\beta_T(T^{-1}(f_n))\}$$
$$= \inf_n \beta_T(h_n)$$

(vii) Since $\{f_n\} \uparrow f$ implies $T^{-1}(f_n) \uparrow T^{-1}(f)$, the result follows using similar arguments as in (vi).

Definition 4.4. We call the triple (X, K_T, β_T) the premeasure space *induced* (or *generated*) by the premeasure space (Y, L, γ).

Definition 4.5. Let (X, K, β) and (Y, L, γ) be premeasure spaces. A mapping $T: X \to Y$ is called *lattice preserving* if $T^{-1}(L) \subseteq K$ and $\beta(T^{-1}(f)) = \gamma(f)$ for all $f \in L$.

Proposition 4.6. *The identity transformation on a premeasure space is lattice preserving.*

Proposition 4.7. *Let* $(X, K, \beta), (Y, L, \gamma)$ *and* (Z, N, θ) *be pre- measure spaces. If* $T_1 : (X, K, \beta) \to (Y, L, \gamma)$ *and* $T_2 : (Y, L, \gamma) \to (Z, N, \theta)$ *are lattice preserving maps, then* $T_2 \, o \, T_1$ *is lattice preserving.*

Proof. For $f \in N, (T_2 \, o \, T_1)^{-1}(f) \in K$, and

$$\beta(T_2 \, o \, T_1)^{-1}(f) = \beta(T_1^{-1}(T_2^{-1}(f))$$
$$= \gamma(T_2^{-1}(f)) = \theta(f)$$, which proves the assertion.

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Volume 2, Issue 1, May 2023
Impact Factor: 7.301

Proposition 4.8. (i) *For a given premeasure space* $(X, \mathcal{K}, \beta)$ *and a map* $T : X \to Y$, *the coinduced premeasure space* $(Y, \mathcal{L}_T, \gamma_T)$ *is the largest premeasure space with respect to which T is lattice preserving* (*i.e. if* $(Y, \mathcal{N}, \theta)$ *be a premeasure space making T lattice preserving, then* $\mathcal{N} \subseteq \mathcal{L}_T$ *and* $\theta(h) = \gamma_T(h)$ *for all* $h \in \mathcal{N}$).

(ii) *For a given premeasure space* $(Y, \mathcal{L}, \gamma)$ *and a map* $T : X \to Y$, *the induced space* $(X, \mathcal{K}_T, \beta_T)$ *is the smallest premeasure space with respect to which T is lattice preserving* (*i.e. if* $(X, \mathcal{N}, \theta)$ *be a premeasure space making T lattice preserving, then* $\mathcal{K}_T \subseteq \mathcal{N}$ *and* $\theta(h) = \beta_T(h)$ *for all* $h \in \mathcal{K}_T$).

Remark 4.9. Let $\mathcal{C}$ be a sublattice of I^X. The definition for a function $\xi : \mathcal{C} \to I$ to be superadditive (subadditive respectively) continue to hold for a real valued function ξ defined on $\mathcal{C}$; Proposition 2.4 is valid for a real valued locally bounded function β.

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