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Some Classes of Univalent Functions Associated with Rusal Differential Operator

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Abstract: *This paper is concerned with the class $\Im(\eta, \xi, \alpha, \beta, \partial, \lambda)$ of normalized analytic univalent functions. We invented Rusal differential operator by making convex combination of Ruschwey and Al-Oboudi differential operator. New subclass $\Im(\eta, \xi, \alpha, \beta, \partial, \lambda)$ is studied with help of Rusal differential operator. Growth theorem, coefficient inequality, convexness and some other interesting properties for given class are examined. Extreme points for the mentioned class are also obtained.*

Mathematics Subject classification: 30C45,30C50

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I. Introduction and Preliminaries

We describe the class $\mathcal{M}$ of all analytic, univalent functions v in the unit disk ℶ = {z: $|z| < 1$} normalized with conditions v (0) =0, v' (0) =1 given by

$v(z) = z - \sum_{k=2}^{\infty} a_k z^k$. (1.1)

[3] has introduced Ruschwey differential operator as bellow,

Definition 1.1. R^n: N→N defined by

$R^n(f(z)) = \frac{z}{(1-z)^{n+1}} \cdot f(z)$ n ∈ ℕ ∪ {0}

$= z + \sum_{k=2}^{\infty} {}^{n+k-1}_{n}C \; a_k z^k$ (z∈U) (1.2)

Where (**.**) is hadmard product defined in (2.3).

We note that R^0 v(z)=f(z), R' v(z)=zv '(z)

[5] and [8] has used following definition 1.2

Definition 1.2. A function v in N is said to be starlike of order α $(0 \le \alpha < 1)$if and only if

Re $\{\frac{zv'(z)}{z}\} > \alpha$ (z∈ U and 0≤ α<1) (1.3)

We write the classes C (0) = C, $CS^*(0) = CS^*$

Definition 1.3. For v ∈ N, [1] has introduced following differential operator, known as Al-Oboudi differential operator.

D^n: N→N defined by

$D^0v(z)=v(z)$ (1.4)

$D^1v(z)=(1-\ )v(z)+\ zv'(z) = D_{□}f(z)$ □≥0. (1.5)

$D^nv(z)=D_{□}(D^{n-1}v(z))$ (1.6)

From (1.5) and (1.6) we have

$D^n(v(z)) = z + \sum_{k=2}^{\infty}[1 + (k-1)\partial]^n a_k z^k$ (z∈U) (1.7)

Jama Salman [7] has introduced the subclass namely $\Im(\eta, \xi, \alpha, \beta, \partial)$ of univalent normalized analytic functions. He studied the geometric properties for the for this subclass. In concern with the given subclass we are extending our further work.

II. Rusal Differential Operator, Class $\mathfrak{I}(\eta, \xi, \alpha, \beta, \partial, \lambda)$.

We formed the Rusal differential operator by making convex combination of Ruschwey & Al-Oboudi differential operators discussed in (1.2) and (1.7) respectively. We also introduced new subclasses $\mathfrak{I}(\eta, \xi, \alpha, \beta, \partial, \lambda)$. which is generalization of $\mathfrak{I}(\eta, \xi, \alpha, \beta, \partial)$ given by [7].

Definition 2.1. For $v_1(z) = z - \sum_{k=2}^{\infty} a_k$ and $v_2(z) = z - \sum_{k=2}^{\infty} a_k$ in $\mathfrak{I}(\eta, \xi, \alpha, \beta, \partial, \lambda)$ defined operation '+' and '*'and '.' as bellow

$$v_1(z) + f_2(z) = (z - \sum_{k=2}^{\infty} a_k z^k) + (z - \sum_{k=2}^{\infty} b_k z^k) = z - \sum_{k=2}^{\infty} (a_k + b_k) z^k \quad (2.1)$$

$$t^* v_1(z) = t * (z - \sum_{k=2}^{\infty} a_k z^k) = z - \sum_{k=2}^{\infty} t a_k z^k \quad (2.2)$$

$$v_1(z) . v_2(z) = (z - \sum_{k=2}^{\infty} a_k z^k) . (z - \sum_{k=2}^{\infty} b_k z^k) = z - \sum_{k=2}^{\infty} (a_k \, b_k) z^k \quad (2.3)$$

Definition 2.2. Let $n \in N \cup \{0\}$, $\lambda \geq 0$, A_λ^n: N→N defined by

$$A_\lambda^n(v(Z)) = (1-\lambda) D^n v(Z) + \lambda R^n v(z). \quad (2.4)$$

On simplifying we observed that,

$$A_\lambda^n(v(z)) = z + \sum_{k=2}^{\infty} ([1 + (k-1)\partial]^n (1-\lambda) + \lambda^{n+k-1}_{\ n}C) \, a_k z^k. \quad (2.5)$$

If n=0, $A_\lambda^0 v(z) = v(z)$.

Definition 2.3. A function v(z) in $\mathcal{M}$ is said to be in $\mathfrak{I}(\eta, \xi, \alpha, \beta, \partial, \lambda)$ if and only if

$$\left| \frac{\frac{z(A_\lambda^n(v))'}{(A_\lambda^n(v)} - 1}{2\xi\left(\frac{z(A_\lambda^n(v))'}{A_\lambda^n(v)} - \alpha\right) - \left(\frac{z(A_\lambda^n(v))'}{A_\lambda^n(v)} - 1\right)} \right| < \beta \quad (2.6)$$

Where $0 \leq < \frac{1}{2\xi}$, $0 < \beta \leq 1$, $\frac{1}{2} \leq \xi \leq 1$, n ε ℕ∪{0}.

We cited the following work which are related with subclass $\mathfrak{I}(\eta, \xi, \alpha, \beta, \partial, \lambda)$.

1. $\mathfrak{I}(0, \frac{1}{2}, 0, 1, \partial, 0)$ is the class of starlike functions
2. $\mathfrak{I}(0, \frac{1}{2}, \alpha, 1, \partial, 0)$ is the class of starlike function of order $0 \leq \alpha < 1$.
3. The class $\mathfrak{I}(0, \xi, \alpha, \beta, 0, 0)$ is the class studied by S.R. Kulkarni [6]
4. The class $\mathfrak{I}(0, \frac{\alpha+1}{2}, 0, \beta, 0, 0)$ is the class studied by Laxminarsimhan [4]
5. The class $\mathfrak{I}(\eta, \xi, \alpha, \beta, \partial, 0)$ is the class studied by S.R. Jumma [7] .

III. Main Results

Our next theorem gives necessary and sufficient condition for the functions in N to be in $\mathfrak{I}(\eta, \xi, \alpha, \beta, \partial, \lambda)$. Also, the corollary (3.2) gives coefficient inequality for functions in the class $\mathfrak{I}(\eta, \xi, \alpha, \beta, \partial, \lambda)$.

Theorem 3.1. v ε $\mathfrak{I}(\eta, \xi, \alpha, \beta, \partial, \lambda)$ if and only if

$$\sum_{k=2}^{\infty} ([1 + (k-1)\partial]^n (1-\lambda) + \lambda^{n+k-1}_{\ n}C)(2\xi\beta(k-\alpha) + (k-1)(1-\beta)) a_k < 2\xi\beta(1-\alpha) \quad (3.1)$$

$n \in \mathbb{N} \cup \{0\}$, $0 < \beta \leq 1$, $0 \leq \alpha < \frac{1}{2\xi}$, $\frac{1}{2} \leq \xi \leq 1$.

Proof. Suppose that,

$$\sum_{k=2}^{\infty} ([1 + (k-1)\partial]^n (1-\lambda) + \lambda^{n+k-1}_{\ n}C)(2\xi\beta(k-\alpha) + (k-1)(1-\beta)) a_k < 2\xi\beta(1-\alpha)$$

$$|z| = 1$$

$$|z (A_\lambda^n f(z))' - A_\lambda^n(f(z))|$$

$$- \beta \,|2\xi(z (A_\lambda^n f(z))' - \alpha A_\lambda^n(f(z)) - (z (A_\lambda^n f(z))' - A_\lambda^n(f(z))|$$

$$= |c - |2\xi\beta(1-\alpha)z + \sum_{k=2}^{\infty} ([1 + (k-1)\partial]^n (1-\lambda) + \lambda^{n+k-1}_{\ n}C)(2\xi\beta(k-\alpha) - \beta(1-k)) a_k z^k|$$

$$\leq |\sum_{k=2}^{\infty} ([1 + (k-1)\partial]^n (1-\lambda) + \lambda^{n+k-1}_{\ n}C)(2\xi\beta(k-\alpha) - \beta(1-k)) a_k - 2\xi\beta(1-\alpha)|$$

$$\leq |\sum_{k=2}^{\infty} ([1 + (k-1)\partial]^n (1-\lambda) + \lambda^{n+k-1}_{\ n}C)(2\xi\beta(k-\alpha) - \beta(1-k)) a_k| - 2\xi\beta(1-\alpha)$$

$\leq 2\,\xi\,\beta(1-\alpha)$ - $2\,\xi\,\beta(1-\alpha)$
= 0.

Hence, by maximum modulus theorem. vε $\Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$.

Conversely suppose that, vε $\Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$.

Definition (2.1) gives us that

$$\left| \frac{\frac{z(A_\lambda^n(v))'}{(A_\lambda^n(v)}-1}{2\xi\left(\frac{z(A_\lambda^n(v))'}{A_\lambda^n(v)}-\alpha\right)-\left(\frac{z(A_\lambda^n(v))'}{A_\lambda^n(v)}-1\right)} \right| < \beta$$

$$\left| \frac{a_k z^{k-1}}{2\xi(1-\alpha)-\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ n}C)(2\xi(k-\alpha)a_k z^{k-1}+\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ n}C)(k-1)a_k z^{k-1}} \right| < \beta$$

But we know that Re{z} ≤ $|z|$

$$\mathrm{Re}\left\{\frac{\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ n}C)(k-1)a_k z^{k-1}}{2\xi(1-\alpha)-\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ n}C)(2\xi(k-\alpha)a_k z^{k-1}+\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ n}C)(k-1)a_k z^{k-1}}\right\}$$

$< \beta$

Letting z→ 1^- through real values in unit disc, we will get

$\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ n}C)(2\xi\beta(k-\alpha)+(k-1)(1-\beta))a_k < 2\,\xi\,\beta(1-\alpha)$

Hence completes the proof.

Moreover, the result is sharp for the functions

$$v(z) = z - \frac{2\xi\beta(1-\alpha)}{([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ n}C)(2\xi\beta(k-\alpha)+(k-1)(1-\beta)}z^k \qquad k \geq 2.$$

Corollary 3.2. Coefficient inequality

Let v ∈ $\Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$ then,

$$a_k \leq \frac{2\xi\beta(1-\alpha)}{2\xi\beta(k-\alpha)+(k-1)(1-\beta)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ n}C)} \tag{3.2}$$

Further we will prove the Growth theorem

Theorem 3.3. Growth theorem

Let the function v(z) defined by (1.1) be in class $\Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$, then

$$|z| - \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)(2\xi\beta(2-\alpha)+(1-\beta))}|z|^2 \leq |v(z)| \leq |z| + \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)(2\xi\beta(2-\alpha)+(1-\beta)}|z|^2 \tag{3.3}$$

Proof: $([1+\partial]^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)(2\xi\beta(2-\alpha)+(1-\beta))\sum_{k=2}^{\infty}a_k \leq$

$\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}_{\ \ n}C)\,(2\xi\beta(k-\alpha)+(k-1)(1-\beta))\,a_k \leq 2\xi\beta(1-\alpha)$

$$\sum_{k=2}^{\infty}a_k \leq \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)(2\xi\beta(2-\alpha)+(1-\beta))}$$

$$|f(z)| = |z-\sum_{k=2}^{\infty}a_k z^k|$$

$$\leq |z|+|z^2|\sum_{k=2}^{\infty}a_k$$

$$= |z| - \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)(2\xi\beta(2-\alpha)+(1-\beta)}|z^2| \leq |f(z)| \tag{3.4}$$

Similarly, we can show that

$$|z| - \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)(2\xi\beta(2-\alpha)+(1-\beta)}|z^2| \leq |f(z)| \tag{3.5}$$

Hence, from (3.4) and (3.5) we can show that

$$|z| - \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)(2\xi\beta(2-\alpha)+(1-\beta)}|z^2| \leq |f(z)| \leq |z| - \frac{2\xi\beta(1-\alpha)}{([1+\partial]^n(1-\lambda)+\lambda^{n+1}_{\ \ n}C)(2\xi\beta(2-\alpha)+(1-\beta)}|z^2|$$

Theorem 3.4. $\Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$ is convex set with respect to operation '+' and '*' defined in (2.1) and (2.2)

Proof: Let $v_1(z)= z- \sum_{k=2}^{\infty} a_k z^k$ and $v_2(z)= z- \sum_{k=2}^{\infty} b_k z^k$ in $\Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$

Define $g(z)= c_1 * v_1(z) + c_2 * v_2(z)$ $\quad c_1+c_2=1$

$= (z - \sum_{k=2}^{\infty} c_1 a_k z^k) + (z - \sum_{k=2}^{\infty} c_2 b_k z^k)$

$= z - \sum_{k=2}^{\infty}(c_1 a_k + c_2 b_k) z^k$

$\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)\,(2\xi\beta(k-\alpha)+(k-1)(1-\beta))\,(c_1 a_k + c_2 b_k) z^k$

$= c_1(\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)\,\big(2\xi\beta(k-\alpha)+(k-1)(1-\beta)\big)a_k) z^k +$

$c_2(\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)\,\big(2\xi\beta(k-\alpha)+(k-1)(1-\beta)\big)b_k) z^k$

$\leq c_1(2\xi\beta(1-\alpha)) + c_2(2\xi\beta(1-\alpha))$

$=2\xi\beta(1-\alpha)(c_1+c_2)$

$=2\xi\beta(1-\alpha)$.

Hence. $\Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$ is convex set.

Theorem 3.5. Let $v_1(z) = z$ and

$$v_k(z)=z-\frac{2\xi\beta(1-\alpha)}{([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)\big(2\xi\beta(k-\alpha)+(k-1)(1-\beta)\big)} z^k \qquad (k\geq 2) \qquad (3.6)$$

Where $n \in \mathbb{N}\cup\{0\}$, $0 < \beta \leq 1$, $0 \leq \alpha < \frac{1}{2\xi}$, $\frac{1}{2} \leq \xi \leq 1$.

Then v (z) is in subclass $\Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$ if and only if $v(z) = \sum_{k=1}^{\infty} \lambda_k v_k(z)$

Where $\lambda_k \geq 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof: Suppose that $v(z) = \sum_{k=1}^{\infty} \lambda_k v_k(z)$

$$= z - \sum_{k=2}^{\infty} \lambda_k \frac{2\xi\beta(1-\alpha)}{([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)\big(2\xi\beta(k-\alpha)+(k-1)(1-\beta)\big)} z^k$$

$$\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)\big(2\xi\beta(k-\alpha)+(k-1)(1-\beta)\big) \times \left(\frac{2\xi\beta(1-\alpha)}{([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)\big(2\xi\beta(k-\alpha)+(k-1)(1-\beta)\big)}\right)\lambda_k$$

$= \sum_{k=2}^{\infty} \lambda_k \; 2\xi\beta(1-\alpha)$

$\leq 2\xi\beta(1-\alpha)$

Hence $v(z) \in \Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$

Conversely assume $v(z) \in \Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$

$$\text{Set } \lambda_k = \frac{2\xi\beta(1-\alpha)([1+(k-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}{}_nC)\big(2\xi\beta(k-\alpha)+(k-1)(1-\beta)\big)}{2\xi\beta(1-\alpha)} a_k \qquad (k \geq 2)$$

$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k$

$\sum_{k=1}^{\infty} \lambda_k v_k(z) = \lambda_1 v_1(z) + \sum_{k=2}^{\infty} \lambda_k v_k(z)$

$= z - \sum_{k=1}^{\infty} a_k z^k$

This completes the proof of theorem.

Theorem 3.6. Suppose $v_i(z) = z - \sum_{k=1}^{\infty} a_{k,i} z^k$ is in the class $\Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$

where, i=1, 2…………t

Define $h(z)= \frac{1}{t} * \sum_{i=1}^{t} v_i(z)$ (3.7)

Then, $h(z) \in \Im(\eta,\xi,\alpha,\beta,\partial,\lambda)$.

Proof. Let $h(z)= \frac{1}{t} * \sum_{i=1}^{t} v_i(z)$

$= \sum_{i=1}^{t} \frac{1}{t} * \boldsymbol{v}_i(z)$

$= \sum_{i=1}^{t} \frac{1}{t} * (z - \sum_{k=1}^{\infty} a_{k,i} z^k)$

$= \sum_{i=1}^{t}(z - \sum_{k=1}^{\infty} \frac{1}{t} a_{k,i} z^k)$

$$= (z - \sum_{k=2}^{\infty} (\sum_{i=1}^{t} \frac{1}{t} a_{k,i}) z^k.$$

$$= z - \sum_{k=2}^{\infty} a_k z^k$$

$$\sum_{k=2}^{\infty}([1+(k-1)\partial]^n(1-\lambda) + \lambda^{n+k-1}{}_{n}C)\left(2\xi\beta(k-\alpha)+(k-1)(1-\beta)\right)\left(\left(\sum_{i=1}^{t}\frac{1}{t} a_{k,i}\right) z^k\right.$$

$$\leq \frac{1}{t}\sum_{i=1}^{t} 2\xi\beta(1-\alpha)$$

$$< 2\xi\beta(1-\alpha)$$

Hence h(z) $\in \Im(\eta, \xi, \alpha, \beta, \partial, \lambda)$.

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