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On Groups with Finitely Many Conjugacy Classes of Non-Normal Subgroups

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Abstract: *The study of finiteness conditions on conjugacy classes is a significant aspect of group theory, given their profound effect on algebraic structures and subgroup dynamics. This study examines groups characterized by a finite number of conjugacy classes of non-normal subgroups, a condition that extends the traditional finiteness concepts associated with the FC-group and T-groups. This study investigated the structural outcomes of this property in both finite and infinite groups. For finite groups, there are notable constraints on subgroup lattices and normality patterns, where a for infinite groups, the focus shifts to those of infinite rank and the importance of the central and FC-Type subgroups. The study also explores the relationships between T-groups and FC-groups, highlighting both commonalities and critical differences. Several examples are provided to illustrate this theory, and a series of open questions are posed to inspire further research.*

Keywords: Conjugacy classes, Non-normal subgroups, FC-group, T-groups, Finite groups, Infinite groups, Subgroups structure

I. INTRODUCTION

The use of finiteness conditions in conjugacy classes has long been a significant method for analyzing the structure of groups. Traditional findings indicate that groups with a limited number of conjugacy classes of elements are either finite or belong to narrowly defined categories, often showing strong centrality and FC-type characteristics. These conditions usually impose strict constraints on the commutator structure and often suggest virtual abelianity or near-nilpotence. Recently, there has been a shift towards more nuanced conjugacy finiteness conditions, where finiteness is applied not universally but to specific families of subgroups. Among these, exploration of groups with a finite number of conjugacy classes of non-normal subgroups has emerged as a natural and conceptually rich extension. Non-normal subgroups represent a breakdown of internal symmetry; thus limiting their conjugacy behavior offers a sensitive measure for examining a group's overall structure. This study was inspired by several independent yet related developments. In the realm of associative algebras, Męcel and Okniński demonstrated that algebras with a finite number of conjugacy classes of left ideals show Strong rigidity and are closely linked to finite representation types [1].

Their research highlighted a recurring theme: the finiteness of conjugacy classes, even when limited to specific substructures, often enforces global algebraic constraints. From a group-theoretic standpoint, Minasyan's foundational work is crucial. Through a series of papers, Minasyan established that groups with a finite number of conjugacy classes of elements are either finite or FC-groups with well-regulated automorphism [2,3].

These findings strongly imply that even weaker finiteness conditions, such as those limited to non-normal subgroups, can result in significant structural implications. An additional motivation comes from the study of subgroup-derived invariants. De Mari and Giovanni explored groups with a finite number of derived non-normal subgroups, revealing that such conditions often necessitate solvability and set limits on the potency class [4].

These findings showed that managing the algebraic behavior within non-normal subgroups can significantly affect the group's overall structure- Barlie and Smith examined related finiteness phenomena by studying groups with a finite number of conjugacy classes of subgroups with a large subnormal defect [5].

This research underscores how limiting conjugacy among "highly non-normal" subgroups imposes strong restrictions on group structure, foreshadowing later developments that focus specifically on non-normal subgroups. More recently, this

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study was extended to contexts involving infinite ranks. De Falco-et.al, analyzed groups of infinite rank with a finite number of conjugacy classes of non-normal subgroups, demonstrating that even in such scenarios, the imposed finiteness condition leads to notable rigidity [6].

Together, these studies reveal that the limitation of having only a finite number of conjugacy classes for non-normal subgroups is a significant constraint. This condition frequently drives groups towards strong centralization, virtual nilpotence, or FC-type behavior, regardless of whether they are finite or infinite. However, a thorough and unified analysis of the groups that meet this criterion is require. In this study, we present several structural insights for groups with a finite number of conjugacy classes of non-normal subgroups. Initially, we demonstrate that in finite groups, this condition imposes strict constraints on the subgroup structure, leading to precise limits on the number and type of non-normal subgroups and often result in solvability or near-nilpotence. For infinite groups, we show that the finiteness of the conjugacy classes of non-normal subgroups necessitates the presence of large normal or central subgroups, establishing strong connections with FC-groups and T-groups. Specifically, we identify broad categories of groups in which every non-normal subgroup has a finite index in its normal closure, and we demonstrate that with some additional mild conditions, the group is either finite-by-abelian or locally nilpotent. We also provide specific examples toiIllustrate the accuracy of our findings and identify situations where the imposed finiteness condition reduces to classical conjugacy finiteness. Finally, we propose several open questions that highlight the gaps and suggest directions for future research in this area.

II. PRELIMINARIES AND DEFINITIONS

Throughout this paper, all groups are assumed to be abstract, and subgroup relations are understood in the standard algebraic sense. The identity elements of Group G are denoted by e.

2.1. Normal Subgroups

Let G be a groups and let $S \leq G$ be a group.

Subgroup S is said to be normal in G, denoted by $S \trianglelefteq G$, if it is invariant under conjugation by every element of G, that is,

$gSg^{-1} = S \quad \text{for all } g \in G$

Equivalently, S is normal G if and only if the left cosets of S in G coincide with the right cosets, that is,

$gS = Sg \quad \text{for all } g \in G.$

2.2. Conjugate Subgroups

Let G be a group and let $S, T \leq G$ be subgroups.

Subgroups S and T are said to be Conjugate in G if there exists an element $g \in G$ such that

$$T = gSg^{-1}.$$

Conjugacy defines an equivalence relation in the set of subgroups of G. The equivalence class of a subgroup S under this relation is called the conjugate class of S.

2.3. Conjugacy Classes of Non-Normal Subgroups

Let G be a group. The conjugate class of subgroups $S \leq G$ is defined as

$S^G = \{gsg^{-1} | g \in G\}.$

Group G is said to have finitely many conjugacy classes of non-normal subgroups if the collection

$\{S^G | S \leq G, S \ntrianglelefteq G.$

is a finite set.

That is, although G may possess infinitely many subgroups, only finitely many distinct conjugate classes arise from the subgroups that are not normal in G.

2.4. FC-Groups

A group G is called an FC-group if every element of G has a finite conjugacy class. Explicity, For each $y \in G$, the set

$$y^G = \{gyg^{-1} | \ g \in G\}$$

is finite.

Equivalently, G is an FC-group if and only if the centralizer

$$C_G(y) = \{g \in G \mid gy = yg\}$$

Has finite index in G for every $y \in G$.

2.5. T-Groups

Group G is called an T-group if normality is a transitive relationship in G
More precisely, G is a T-group if whenever
$S \trianglelefteq T$ and $T \trianglelefteq G$,
it follow that
$S \trianglelefteq G$.
Thus in a T-group, every subgroups that is normal in the normal subgroup of G is necessarily normal in the G itself.

III. BASIC OBSERVATIONS

In this section, we present several direct outcomes of the premise that a group has a finite number of conjugacy classes of non-normal subgroups. While basic, these observations help define the limits of the finiteness condition and emphasize the scenarios in which it becomes limiting.

3.1. Trivial validity in the Absence of Non-Normal Subgroups

If all subgroups of group G are normal,G automatically meets the finiteness condition. Indeed, this definition pertains solely to conjugacy classes that stem from non-normal subgroups. In the absence of such subgroups, the related collection of conjugacy classes is empty and, thus, finite. This situation is found, for instance, in abelian groups and, more broadly, in dedekind groups. Therefore, the finiteness condition is fulfilled in these instances.

3.2. Groups with Finitely Many Subgroups

While every finite group has finitely many subgroups, the present condition is nontrivial because it restricts the distribution and sizes of the conjugacy classes of the non-normal subgroup.

3.3. Structural Restrictions in Infinite Groups

The finiteness of non-normal conjugacy classes becomes very limite when G is infinite. Infinite groups generally have infinitely many non-normal subgroups that- often belong to infinitely many distinct conjugacy classes. The finiteness condition prevents this from occurring and requires all non-normal subgroups to fall into a finite number of conjugacy types. Consequently, non-normality is greatly restricted, and the subgroup structure of G shows a strong regularity. This restriction often appears as large normal or central subgroups, FC-type behavior, or normal by finite phenomena. As a result, infinite groups meeting this condition frequently resemble abelian, nilpotent, or finite abelian groups.

Finite Groups:

In the realm of finite groups, the condition that a group is limited to a finite number of conjugacy classes of non-normal subgroups imposes significant restrictions on leadings to profound structural outcomes. Although every finite group has a finite number of subgroups, this requirement transcends a simple numerical constraint; it dictates the manner in which conjugation affects non-normal subgroups and, consequently, influences the pattern of asymmetry within the group.

For a finite group G, the size of the conjugacy class of subgroup $S \leq G$ is given by

$$|S^G| = [G: N_G(S)],$$

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Where $N_G(S)$ denotes the normalizer. Therefore, limiting the number of conjugacy classes of non-normal subgroups necessitates a limit on the normalizer indices of the normalizers belonging to a finite set. This results in a consistent limit on the normalizer indices, indicating that the non-normal subgroup must possess relatively large normalizers. Consequently, their conjugates formed normal subgroups with bounded indices, guaranteeing the existence of substantial normal substructures within G.

The impact of this phenomenon on subgroup lattices was significant. Typically, finite groups display complex lattices with numerous non-normal subgroups in various configurations. However, when finiteness is considered, this complexity is diminished: non-normal subgroups are restricted to a finite number of controlled conjugacy types, and most subgroup variations are confined to the normal subgroup lattice.

A particularly significant implication is the Sylow subgroups. A finite group is nilpotent if and only if all its Sylow subgroups are normal, meaning that the condition of finiteness either enforces nilpotency or significantly restricts the behavior of non-normal Sylow subgroups in finite groups. Even if nilpotency is not achieved, non-normal Sy;low subgroup must belong to a finite number of conjugacy classes and generate large normal closures, which greatly limits the extent of non-nilpotent behavior within the group.

In contrast to the infinite case, where additional assumptions, such as finite rank, FC-type conditions, or centrality hypotheses, are often required, finiteness alone is sufficient to exert substantial control over finite groups. Consequently, finite groups that satisfy this condition are often nilpotent, supersolvable, or similar to Dedekind groups, with non-normality confined to a small and well-structured portion of the subgroup lattice.

Theorem 4.1 (Finite Case: Dichotomy and Lattice Constraint)

Let G be a finite group with finitely many conjugacy classes of non-normal subgroups. Then exactly one of the following holds:

1. Nilpotent case: G is nilpotent

2. Bounded non-normality case:

There exists a constant $M \in \mathbb{N}$ such that for every non-normal subgroup $S \leq G$,

$$[G: N_G(S)] \leq M.$$

In this case, the set of non-normal subgroups is finite up to conjugacy, and the subgroup lattice of G is highly constrained. Moreover, under additional mild assumptions (such as solvability or restrictions on the Sylow subgroup), the group in case (2) is frequently supersolvable or close to dedekind groups.

Lemma 4.1 (Normalizer-Conjugacy Correspondence)

Let G be a finite group and let $S \leq G$. Then $|S^G| = [G : N_G(S)]$.

Proof: Group G acts on a set of subgroups of G via conjugation. The stabilizer of S under this action was precisely $N_G(S)$. Using the Orbit-Stabilizer Theorem,

$|S^G| = [\, G : N_G(S)]$.

Lemma 4.2 (Uniform Boundedness of Normalizer Indices)

Let G be a finite group with finitely many conjugacy classes of non-normal subgroups. Thus there exists a constant $M \in \mathbb{N}$ such that

$$[G: N_G(S) \leq M$$

For every non-normal subgroup $S \leq G$.

Proof: By hypothesis, the set

$$C = \{S^G \mid S \leq G, S \ntrianglelefteq G\}$$

is finite.

According to Lemma 4.1, each conjugacy class S^Ghas size$[G : N_G(S)]$. Because C is finite, the set of indices is finite. Hence there exist

$$M = \max\{[G : N_G(S)] \mid S \leq G, S \ntrianglelefteq G\}.$$

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Lemma 4.3 (Nilpotent Groups Satisfy the Condition Trivially)
If G is a finite nilpotent group, then it has no non-normal subgroups. In particular, there are finitely many conjugacy classes of non-normal subgroups.
Proof: A finite group is nilpotent if and only if it is the direct product of its Sylow subgroups. In such groups, every subgroup was subnormal, and in finite nilpotent groups, all sylow subgroup are normal. Hence all subgroups of G are normal, and the set of conjugacy classes of the non-normal subgroups is empty.

Lemma 4.4 (Existence of Non-Normal Subgroups in the Non-Nilpotent Case)
Let G be a finite group that is non-nilpotent. G contained at least one non-normal subgroup.
Proof: If every Sylow subgroup of G is normal, then G would be nilpotent. Because G is not nilpotent, at least one Sylow subgroup is non-normal.

Lemma 4.5 (Constraint on Normal Closures)
Let G be finite and let $S \leq G$ be a non-normal subgroup satisfying

$$[G : N_G(S)] \leq M.$$

Then the normal closure $< S^G >$ is generated by most M conjugates of S and bounded index in G.
Proof: By Lemma 4.1,

$$|S^G| = [G : N_G(S)] \leq M.$$

Thus the normal closure $< S^G >$ is generated by a most M conjugates of S.
Because each conjugate corresponds to a coset of $N_G(S)$, the index of $< S^G >$ in G is bounded in terms of M.
Lemma 4.6 (Restriction on Sylow Subgroups)
Let G be a finite group that satisfies the bounded normalizer condition. Then only finitely many Sylow subgroups of G can be non-normal, and each such Sylow subgroup has a normalizer bounded index.
Proof: Let P be a non-normal Sylow p-subgroup in G.

$$[G : N_G(S)] \leq M = |P^G| \leq M.$$

Hence the number of P conjugates is bounded. Since distinct prime give rise to distinct Sylow subgroups, this bounds both the number and behavior of non-normal Sylow subgroups.

Proof of theorem 4.1
Let G be a finite group with finitely many conjugacy classes of non-normal subgroups.
If all subgroups of G are normal, then G is nilpotent according to Lemma 4.3 and case (1) holds.
Otherwise, by Lemma 4.4, G contains at least one non-normal subgroup.
By Lemma 4.2,

There exist a constant $M \in \mathbb{N}$ such that
$[G : N_G(S)] \leq M$ for all $S \leq G$ with $S \ntrianglelefteq G$.
Lemmas 4.5 and 4.6 show that non-normal subgroups generate large normal subgroups and that non-normality is uniformly bounded throughout the subgroups lattice. This establishes case (2).
Infinite Groups:
In contrast to finite groups, the condition in which an infinite group possesses only a finite number of conjugacy classes of non-normal subgroups is highly restrictive. Typically, infinite groups have an abundance of non-normal subgroups, that are distributed among numerous distinct conjugacy classes. Therefore, limiting these classes to a finite number imposes significant global regularity and significantly restricts possible forms of non-normality.
A major consequence of this condition is that non-normal subgroups must exhibit uniform conjugacy. Specifically, for any non-normal subgroup $S \leq G$, conjugacy class

$$S^G = \{gSg^{-1} \mid g \in G\}$$

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Must belong to a finite set of classes. Importantly, the size of S^G is limited regardless of the choice of S.
This directly implies that the normal closure $< S^G >$ is generated by a finite number of conjugates of S. making it a finitely generated normal subgroup of G. Consequently, while infinite groups generally display complex and limitless conjugacy dynamics, the condition of finiteness imposes strict rigidity on all non-normal subgroups.

5.1 Normal closures and Finite Conjugacy

Let G be an infinite groups with a finite number of conjugacy classes of non-normal subgroups. Then, for each non-normal subgroup $S \leq G$, conjugacy class S^G is finite and normal closure $< S^G >$ is generated by a finite number of conjugates of S.
Moreover, the index

$$[G : N_G(S)] = |\, S^G \,|$$

is bounded by a constant that does not depend on S. Consequently, the normalizer indices of all non-normal subgroups were uniformly limited, and the extent to which a subgroup was not normal was consistently controlled throughout the subgroup lattice.
This situation starkly contrasts with the typical behaviour of infinite groups, where non-normal subgroups may have an number of conjugates, infinitely generated normal closures, and highly irregular conjugacy patterns. The requirement of finiteness curtails such complexity and ensures that non-normality occurs in a strictly regulated and predictable manner.

Theorem 5.1 (Uniform Boundedness of Conjugacy and Normal Closures in infinite Groups)
Let G be an infinite group with finitely many conjugacy classes of non-normal subgroups. Then there exists a constant $M \in \mathbb{N}$ such that for every non-normal subgroup $S \leq G$,

$$|\, S^G \,| = [G : N_G(S)] \leq M.$$

Consequently, the normal closure $< S^G >$ is generated by at most M conjugates of S and is a finitely generated normal subgroup of G. In particular, the degree of non-normality of the G subgroup was uniformly bounded.

Proof: By hypothesis, the set

$$C = \{S^G \,| S \leq G, S \ntrianglelefteq G\}$$

Of conjugacy classes of non-normal subgroups of G is finite.

For each non-normal subgroup $S \leq G$, group G acts by conjugation on the set of subgroups of G, and the stabilizer of S under this action is the normalizer $N_G(S)$. Using the Orbit-Stabilizer Theorem, we have

$$|\, S^G \,| = [\, G{:}\, N_G(S)].$$

Since C is finite, the set of indices

$$\{\, [\, G : N_G(S)] | S \leq G, S \ntrianglelefteq G\}$$

is finite. Hence there exists a constant

$$M = \max\{\, [G : N_G(S)] \,| S \leq G, S \ntrianglelefteq G < \infty,$$

Which proves the first claim.
Now fix a non-normal subgroup $S \leq G$. Since $|\, S^G \,| \leq M$, the normal closure

$$< S^G >= < gSg^{-1} \,|\, g \in G >$$

is generated by at most M conjugates of S. Therefore, $< S^G >$ is a finitely generated normal subgroup of G.

This uniform bound shows that the non-normal subgroup of G cannot exhibit an arbitrarily large or irregular conjugacy behaviour. Thus, the failure of normality in G is Quantitatively controlled throughout the subgroup lattice, thereby completing the proof.

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Corollary 5.2 (Finite-Rank Rigidity)

Let G be an infinite group with finitely many conjugacy classes of non-normal subgroups. If G has finite rank (equivalently, there exists $r \in \mathbb{N}$ such that every finitely generated subgroup of G can be generated by at most r elements), then G is either

1. finitely-by-abelian or
2. locally nilpotent.

Proof: By Theorem 5.1, there exists a constant $M \in \mathbb{N}$ such that for every $\mathrm{non-normal\ subgroup}\ S \leq G,$

$$| S^G | = [G : N_G(S)] \leq M,$$

The normal closure $< S^G >$ is generated by at most M conjugates of S. Each normal closure is a finitely generated normal subgroup of G.

Let H be any finitely generated subgroup G. If H is normal in G, then there is nothing to prove. Suppose Instead suppose that H is non-normal. Then its normal Closure $< H^G >$ is finitely generated by M conjugates of H at most. Because G has a finite rank, there a uniform bound exists on the rank of all such normal closures.

This implies that the family of normal closures of non-normal finitely generated subgroups is uniformly generated. Consequenty, the derived subgroup G' is constrained to lie inside a finitely generated normal subgroup of a finite rank. Standard results for finite-rank groups then imply that G' is locally finite or nilpotent.

If G' is finite, then G is finite-by-abelian.

If G' is infinite, the finite-rank condition forces G' to be locally nilpotent hence, G itself is locally nilpotent.

Therefore, under the finite-rank assumption, the finiteness of conjugacy classes of non-normal subgroups forces G to be either finite-by-abelian or locally nilpotent, as claimed.

VI.Relation with T-Groups and FC-Groups:

This paper examines a finiteness condition that position the group class with a limited number of conjugacy classes of non-normal subgroups between two extensively researched categories in group theory: the FC-group and T-groups. In this section, we elucidates these connections and emphasize both commonalities and key distinctions.

6.1 Relation with FC-Groups

Group G is called an FC-group if every elements of G has a finite conjugacy class. Equivalently,

$[G : C_G(x)] < \infty$ for all $x \in G$.

It is well known that FC-groups possess strong properties; in particular, their derived subgroup is locally finite and the group is finite-bby-abelian [2,3].

The finiteness condition investigated in this study is limited in scope. Instead of controlling the conjugacy classes of elements, it restricts the conjugacy classes of non-normal groups. closely analogous to that of Fc groups at the subgroup level.

Indeed, if G has finitely many conjugacy classes of non-normal subgroups, for every non-normal subgroup$S \leq G$, the index

$$[G : N_G(S)]$$

The group was consistently limited, which means that the manner in which G acts on its non-normal subgroups through conjugation was significantly restricted. This result in the normal closures being finitely generated into normal subgroups of G.

Therefore, even though the group itself might not qualify as an FC group, it frequently includes a substantial FC subgroup or is characterized as FC-by-finite. Specifically, non-normality cannot manifest in an unrestricted or disorderly manner, and conjugation tends to act almost centrally beyond finite obstruction.

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6.2 Relation with T-Groups

Group G is called a T-group if normality is a transitive relation in G; that is
Whenever

$$S \trianglelefteq T \trianglelefteq G,$$

It follows that

$$S \trianglelefteq G.$$

T-groups are known to have strongly constrained subgroup lattices and very limited non-normal behavior.

When the condition of having a limited number of conjugacy classes is imposed for the non-normal subgroup, rigidity becomes even more evident. Consider G as a T-group with a finite number of conjugate classes for non-normal subgroups. In this scenario, the non-normal subgroup cannot form extensive chains or pass through intermediate normal subgroups, as such an expansion would violate either the transitivity of normality or the finite nature of conjugacy classes.

As a result, a T-group that meets the finiteness criterion closely mirror the dedekind behavior. Specifically, all subgroups of G are normal or the collection of non-normal subgroups is restricted. In both situations, non-normality was tightly controlled and occured in a highly regulated manner. Notably, many such phenomena emerge as finite extensions of T groups.

Examples:

In this section, we present examples of the results obtained in previous sections. These examples demonstrate that the finiteness condition for conjugacy classes of the non-normal subgroup is non-trivial, applies to a broad rang of groups, and is strictly weaker than several classical finiteness conditions.

7.1 Trivial and Boundary Cases

Example 7.1 (Abelian Groups)

Every abelian group satisfies the finiteness condition trivially, as all its subgroups are normal. Consequently, there are no non-normal subgroups, and hence, no conjugacy classes of non-normal subgroups.

Example 7.2 (Dedekind Groups)

Generally, Dedekind Groups that is, groups in which every subgroups is normal, also trivially satisfy the finiteness condition. This class includes Hamiltonian groups, such as

$$Q_8 \times A,$$

Where A is an abelian group.

These groups illustrate that the finiteness condition does not force abelianity.

7.2 Finite Non-Nilpotent Examples

Example 7.3 (The Symmetric Group S_3)

Let $G = S_3$. G is non-abelian and non-nilpotent. however, it has only finitely many non-normal subgroups, all belonging to finitely many conjugacy classes.

In particular

The Sylow 3-subgroup is normal, and

The Sylow 2-subgroups form a single conjugacy class.

Thus, S_3 satisfies the hypothesis of Theorem 4.1 in the bounded non-normality case. This example of nilpotence is not forced by the finiteness condition of finite groups.

Example 7.4 (Dihedral Groups)

Let $= D_{2p}$, where p is odd prime. Then G has a finite number of non-normal subgroups, all of which lie withiin many finite conjugacy classes.

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7.3 Infinite Examples

Example 7.5 (FC-Group)

Every FC-group with finitely many conjugacy classes of non-normal subgroups satisfies the hypotheses of Theorem 5.1. Typical examples include finite by abelian groups.

Open Problems:

The limitation on the number of conjugacy classes of non-normal subgroups places significant structural limitations on both the finite and infinite groups. Nonetheless, several key questions remain to be answered.

Classification

Identify finitely generated infinite groups that have a limited number of conjugacy classes of non-normal subgroups. Specifically, do such groups necessarily have to be finite-by-abelian or locally nilpotent without additional rank conditions? This issue is driven by established rigidity findings for FC-groups and groups with a few non-normal subgroups. Growth and rigidity. Does the combination of polynomial growth and this finiteness condition require virtual nilpotence? How does this condition affect the subgroup growth and geometric invariants? This inquiry links the current framework with the traditional Gromov-type results and subgroup growth theory.

Infinite simple groups

Are there infinitely simple groups that meet this finiteness condition? If not, what barriers does simplicity create? Similar phenomena were observed in a study of groups with limited non-normal subgroups and restricted conjugacy behavior.

Automorphism groups

What limitations do this condition place in automorphism group Aut(G)? Specifically, must Aut(G) be finite-by-abelian or FC-type? This question is inspired by findings on automorphism of FC-groups and groups with finite conjugacy conditions.

Quantitative and algorithmic aspects.

Can precise limits be established that relate subgroup invariants to the number of conjugacy classes in non-normal subgroups? Is this finiteness condition determinable for the finitely presented groups? Similar quantitative challenges arise in studies of bounded conjugacy classes and subgroup lattices.

Further extensions

Expand the theory to include profinite or locally compact groups, and investigate parallels with ideal conjugacy in group algebras, as inspired by the work of Mecel and Okniński [1].

Conclusion:

We introduced and examined groups with a finite number of conjugacy classes of non-normal subgroups, a finiteness condition that naturally extends the traditional conjugacy constraints in group theory. This property conceptually sits between FC-groups, which manage the conjugacy of elements, and T-groups, in which normality shows strong transitivity.

For finite groups, this condition either enforces nilpotence or restricts non-normal subgroups to a uniformly bounded and highly organized section of the subgroup lattice, which is consistent with previous findings on groups with few non-normal subgroups.

In the case of infinite groups, it results in strong rigidity: non-normal subgroups have uniformly bounded conjugacy classes and finitely generated normal closures, leading to a finite-by-abelian or locally nilpotent structure under mild additional conditions, mirroring phenomena seen in FC-groups and related classes.

These findings demonstrate that limiting conjugacy at the subgroup level has significant implications for the overall group structure and opens several promising avenues for further research.

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