

A Review of Applications of Fractional Differential Equations in Digital Signal Processing

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Abstract: Fractional differential equations have emerged as an important mathematical framework for modelling systems that exhibit memory, hereditary behaviour, nonlocal interactions, and frequency-dependent characteristics. Unlike classical differential equations, which generally describe systems through integer-order derivatives, fractional differential equations employ derivatives of non-integer order and provide additional flexibility in representing complex dynamic processes. In digital signal processing, these properties are particularly useful for designing fractional-order filters, improving noise reduction, modelling signals with long-range dependence, performing image and speech enhancement, analysing biomedical signals, and developing adaptive filtering algorithms.

Keywords: Fractional Differential Equations, Fractional Calculus, Digital Signal Processing, Fractional-Order Filters.

I. INTRODUCTION

Digital signal processing is concerned with the mathematical representation, analysis, transformation, filtering, and interpretation of discrete-time signals. It plays a fundamental role in telecommunications, audio engineering, radar systems, biomedical instrumentation, image analysis, control systems, and sensor networks. Conventional digital signal processing techniques are commonly developed using integer-order differentiation, integration, linear difference equations, and the discrete Fourier transform. Although these methods have achieved considerable success, some practical signals and physical systems exhibit memory effects, nonlocal dependence, anomalous diffusion, power-law decay, and frequency-dependent behaviour that may be difficult to represent compactly using conventional integer-order models.

Fractional calculus extends classical calculus by allowing differentiation and integration to orders that are not restricted to integers. Its mathematical foundations can be traced to early developments in the theory of non-integer differentiation, while modern formulations have established systematic methods for defining fractional derivatives and integrals (Podlubny, 1999; Samko et al., 1993). A fractional derivative can represent a system response that depends not only on the present input but also on its past history. This property makes fractional calculus relevant to the modelling and processing of signals arising from systems with long-term dependence or distributed temporal dynamics. In digital signal processing, fractional-order operators can be approximated by discrete-time algorithms and implemented through finite impulse response (FIR) filters, infinite impulse response (IIR) filters, recursive approximations, and rational transfer functions. Such implementations allow fractional differentiation and integration to be incorporated into digital filtering and system analysis. Fractional-order techniques can provide additional degrees of freedom in filter design, enabling designers to adjust magnitude and phase responses beyond those available from a particular conventional filter structure (Monje et al., 2010).

A related development is the fractional Fourier transform, which generalises conventional Fourier analysis by introducing a continuous transformation order. It offers a representation between the time and frequency domains and is useful for analysing signals with chirp-like structures and time-frequency characteristics (Almeida, 1994; Ozaktas et al., 2001). Although the fractional Fourier transform and fractional differential equations are mathematically distinct

concepts, both belong to the broader field of fractional-order signal analysis and are relevant to modern signal-processing research.

The importance of fractional-order methods is not limited to theoretical modelling. Their potential applications include speech enhancement, noise suppression, biomedical signal analysis, edge detection, adaptive filtering, communication systems, and the analysis of nonstationary signals. Nevertheless, fractional-order operators often require more computation and memory than simple integer-order alternatives. Their practical effectiveness depends on the selected derivative definition, numerical approximation, sampling frequency, filter order, stability requirements, and characteristics of the signal being processed.

The purpose of this review is to examine the mathematical principles, major applications, implementation methods, advantages, limitations, and future prospects of fractional differential equations in digital signal processing. It also distinguishes direct applications of fractional differential operators from related techniques, such as fractional Fourier analysis, to provide a coherent foundation for further research.

OBJECTIVES OF THE REVIEW

The principal objectives of this review are:

- To explain the mathematical foundations of fractional differential equations relevant to digital signal processing.
- To examine the principal numerical methods used to implement fractional-order operators in discrete-time systems.
- To review applications in digital filtering, noise reduction, speech processing, biomedical signal analysis, and time-frequency analysis.
- To compare fractional-order methods with conventional integer-order signal-processing techniques.
- To identify computational, numerical, and practical limitations affecting real-time implementation.
- To suggest future research directions for developing efficient and reliable fractional-order DSP systems.

REVIEW METHODOLOGY

This paper adopts a narrative literature-review approach to examine the mathematical foundations and applications of fractional differential equations in digital signal processing. The literature considered includes foundational books on fractional calculus, research articles on fractional signals and systems, studies of fractional-order digital filters, and publications addressing fractional Fourier analysis and biomedical signal processing.

The review focuses on three interconnected areas: (i) theoretical development of fractional differential operators, (ii) computational implementation in discrete-time signal-processing systems, and (iii) practical applications in filtering, denoising, signal enhancement, and time-frequency analysis. Foundational contributions by Podlubny (1999), Samko et al. (1993), and Monje et al. (2010) provide the mathematical basis, while research by Magin et al. (2011), Ferdi (2012), Muresan et al. (2021), and Ortigueira (2024) informs the discussion of signal-processing applications and implementation approaches.

The literature is synthesised thematically rather than through a quantitative meta-analysis. The paper does not claim to provide an exhaustive systematic search, and no new experimental dataset or original performance measurements are introduced. Consequently, comparisons of computational cost and filtering performance are qualitative unless explicitly supported by a mathematical expression or a cited study.

MATHEMATICAL FOUNDATIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

1. Concept of fractional calculus

Fractional calculus generalises ordinary differentiation and integration to arbitrary real or complex orders. In classical calculus, the first derivative describes the instantaneous rate of change, while the second derivative describes the rate of change of the first derivative. Fractional calculus extends these concepts to an order α , where α need not be an integer. A general fractional derivative can be represented as

$$D^{\alpha}x(t), \quad \alpha \in \mathbb{R}^{+}.$$

When $\alpha = 1$, the operator corresponds to ordinary first-order differentiation. When $\alpha = 2$, it corresponds to second-order differentiation under the standard definitions. For non-integer values, such as $\alpha = 0.5$, the operator represents a half-order derivative.

Fractional operators are especially useful in systems where the current response depends on a weighted history of previous observations. This history-dependent behaviour distinguishes many fractional models from conventional differential equations involving only local integer-order derivatives (Podlubny, 1999; Samko et al., 1993).

2. Riemann–Liouville fractional derivative

One commonly used definition is the Riemann–Liouville fractional derivative. For a function $x(t)$, a lower terminal a , and an order α satisfying $n - 1 < \alpha < n$, it is defined as

$${}_a D_t^{\alpha} x(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{x(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau.$$

Here, n is an integer, and $\Gamma(\cdot)$ denotes the gamma function. The integral incorporates past values of the function, which gives the operator a nonlocal character.

In signal processing, this property is relevant when a signal's behaviour cannot be adequately described using only its instantaneous value and a finite number of ordinary derivatives. However, the Riemann–Liouville derivative can require initial conditions expressed in terms of fractional integrals or derivatives, which may be less convenient for physical systems specified by conventional initial values.

3. Caputo fractional derivative

The Caputo derivative is another important formulation, particularly for engineering applications where initial conditions are measured in terms of ordinary derivatives. It is defined by

$${}_a^C D_t^{\alpha} x(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{x^{(n)}(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau, \quad n - 1 < \alpha < n.$$

Unlike the Riemann–Liouville definition, the Caputo formulation differentiates the function before integrating it. For sufficiently regular functions, this allows initial conditions to be expressed using ordinary integer-order derivatives.

The Caputo derivative is therefore useful in the modelling of physical systems and in the formulation of fractional differential equations whose initial states have direct physical meaning. The choice between Caputo and Riemann–Liouville derivatives depends on the model, the available initial conditions, and the intended interpretation of the system (Podlubny, 1999).

4. Grünwald–Letnikov fractional derivative

For discrete-time implementation, the Grünwald–Letnikov definition is particularly important because it naturally leads to a weighted finite-difference approximation. A commonly used form is

$$D^{\alpha}x(t) = \lim_{h \rightarrow 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor (t-a)/h \rfloor} (-1)^k \binom{\alpha}{k} x(t - kh),$$

where h is the sampling interval and

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)}.$$

In a digital system, the continuous-time limit is replaced by a finite sampling interval T_s , and the sum is truncated to a finite number of past samples. The resulting approximation is computationally implementable and provides a direct connection between fractional calculus and digital filtering.

5. Fractional-order differential equation

A general linear fractional differential equation may be written as

$$\sum_{k=0}^N a_k D^{\alpha_k} y(t) = \sum_{m=0}^M b_m D^{\beta_m} x(t),$$

where $x(t)$ is the input, $y(t)$ is the output, a_k and b_m are model coefficients, and α_k and β_m are derivative orders.

Under appropriate assumptions concerning the derivative definitions and initial conditions, its zero-initial-condition transfer-function representation is

$$H(s) = \frac{\sum_{m=0}^M b_m s^{\beta_m}}{\sum_{k=0}^N a_k s^{\alpha_k}}.$$

The variable s is the Laplace-domain complex frequency. Fractional powers of s provide additional freedom for shaping the frequency response, although their practical digital implementation generally requires approximation over a specified frequency range.

DISCRETE-TIME IMPLEMENTATION OF FRACTIONAL OPERATORS

1. Fractional difference equation

A fractional-order digital differentiator can be constructed from the Grünwald–Letnikov approximation. For a sampled signal $x[n]$, one practical finite-memory implementation is

$$y[n] \approx \frac{1}{T_s^\alpha} \sum_{k=0}^L c_k x[n - k],$$

where L is the memory length and the coefficients are

$$c_k = (-1)^k \binom{\alpha}{k}.$$

The coefficients can also be generated recursively:

$$c_0 = 1, \quad c_k = \left(1 - \frac{\alpha + 1}{k}\right) c_{k-1}.$$

This recursive expression avoids repeated evaluation of gamma functions. The output at each sample is computed from the current input and a weighted collection of preceding samples.

For $\alpha = 1$, the coefficients reduce to the familiar first-order difference pattern. For non-integer α , the operator incorporates a longer sequence of coefficients, thereby approximating fractional differentiation.

The truncation length L determines the amount of signal history retained. Increasing L can improve the approximation over a target frequency range but also increases memory usage and computational cost. Consequently, the memory length must be selected according to the sampling rate, desired accuracy, latency, and available processing resources.

2. Z-transform representation

The finite-memory fractional differentiator in Equation has the transfer function

$$H(z) = \frac{1}{T_s^\alpha} \sum_{k=0}^L c_k z^{-k}.$$

This expression is a finite impulse response approximation. Its finite impulse response structure makes implementation straightforward, but the approximation does not exactly reproduce an ideal fractional derivative over all frequencies.

An alternative uses the formal generating function

$$H(z) = \frac{1}{T_s^\alpha} (1 - z^{-1})^\alpha.$$

For a non-integer α , the binomial expansion contains infinitely many terms. Practical implementations must therefore truncate the expansion or approximate the fractional power by a rational function.

Equation is useful for theoretical analysis and discrete fractional differencing, but finite-memory truncation and frequency-response errors must be considered in real systems.

II. CONCLUSION

Fractional differential equations provide a flexible mathematical framework for extending conventional digital signal processing. By incorporating non-integer-order differentiation and integration, they enable the modelling of systems with memory effects, power-law behaviour, and complex frequency-dependent dynamics. The Riemann–Liouville, Caputo, and Grünwald–Letnikov formulations provide alternative mathematical descriptions of fractional derivatives, while discrete-time approximations and rational transfer functions enable their implementation in digital systems.

The literature reviewed in this paper demonstrates the relevance of fractional-order techniques to digital filtering, signal denoising, biomedical signal processing, speech enhancement, adaptive filtering, and the analysis of long-memory signals. Fractional-order methods may provide additional control over filter characteristics and may be especially useful when conventional integer-order models do not adequately represent the behaviour of the system under study. The fractional Fourier transform further extends signal-analysis capabilities by offering alternative time-frequency representations, although it should not be confused with fractional differential operators.

At the same time, fractional-order processing introduces important practical challenges. Infinite-memory behaviour must be approximated for digital implementation, and the accuracy of the resulting algorithm depends on memory length, approximation method, sampling frequency, and numerical precision. Increased mathematical flexibility also does not guarantee improved performance in every application. Accordingly, fractional-order filters should be

compared against established alternatives using appropriate datasets, controlled experimental conditions, and objective performance measures.

In conclusion, fractional differential equations should be viewed as a complementary extension of conventional digital signal processing rather than a universal replacement for integer-order techniques. Their long-term contribution will depend on the development of computationally efficient implementations, robust parameter-estimation methods, reproducible evaluation procedures, and application-specific validation. Continued research in these areas can strengthen the role of fractional calculus in modern signal analysis and digital system design.

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