

Applications of Fractional Differential Equations in Image Processing and Enhancement

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Abstract: Fractional differential equations have emerged as a promising mathematical framework for image processing and enhancement because they extend classical differential equations by incorporating derivatives of non-integer order. Unlike conventional integer-order operators, fractional-order operators can represent nonlocal interactions and long-range dependencies, which are useful for modelling complex image structures, preserving edges, reducing noise, and enhancing contrast. This research paper examines the mathematical foundations and practical applications of fractional differential equations in digital image processing. Particular attention is given to fractional-order diffusion, image denoising, edge preservation, image sharpening, contrast enhancement, and image restoration

Keywords: Fractional Differential Equations, Fractional Calculus, Image Processing, Image Enhancement, Fractional Diffusion

I. INTRODUCTION

Digital image processing is an essential area of computational mathematics, computer science, engineering, medical imaging, remote sensing, computer vision, and scientific visualization. Digital images frequently contain noise, low contrast, blurred boundaries, illumination variations, and loss of fine structural information. These limitations may reduce the accuracy of subsequent tasks, such as object recognition, segmentation, classification, and feature extraction. Therefore, developing mathematical techniques that improve image quality while preserving important visual information remains a significant research objective.

Classical image-processing methods frequently employ integer-order differential operators, such as the gradient and Laplacian, to detect edges, smooth intensity variations, and enhance image details. Partial differential equations (PDEs) provide a systematic framework for describing the evolution of image intensity over time. For example, the heat equation can be used for image smoothing, while nonlinear diffusion models can reduce noise while attempting to preserve meaningful edges. Perona and Malik (2010) introduced an influential nonlinear diffusion approach that adapts smoothing to local image gradients, demonstrating the potential of PDE-based methods for edge-preserving image processing.

Despite their usefulness, classical differential operators have limitations. Integer-order operators primarily describe local variations, and their behaviour may not adequately represent certain long-range dependencies or complex spatial structures. Excessive smoothing can blur edges and remove fine textures, whereas aggressive sharpening can amplify noise. These difficulties motivate the investigation of more flexible mathematical operators.

Fractional calculus generalizes differentiation and integration to non-integer orders. Its development includes the Riemann–Liouville and Caputo formulations, which provide different ways to define fractional derivatives and incorporate initial conditions (Podlubny, 2012). In image processing, fractional-order operators can be constructed to account for broader spatial interactions and to adjust the response of a filter to different image frequencies.

Fractional differential equations can consequently be applied to image denoising, enhancement, restoration, edge detection, and texture analysis. For example, fractional diffusion can modify the rate at which image structures evolve,

while fractional-order spatial filters can adjust the amplification or attenuation of image frequencies. These properties may be valuable in medical images, satellite images, microscopy, and low-light photography, where both noise reduction and preservation of fine details are important.

The present paper examines the theoretical foundations, mathematical models, applications, and computational challenges of fractional differential equations in image processing and enhancement. It aims to connect the mathematical theory of fractional operators with practical image-processing tasks and to identify opportunities for future research.

REVIEW OF LITERATURE

The application of fractional calculus in image processing has developed through research on fractional-order filters, diffusion equations, variational models, and computer vision.

Yang et al., (2016) presented a comprehensive review of fractional calculus in image processing, covering image enhancement, denoising, edge detection, segmentation, registration, recognition, fusion, encryption, compression, and restoration. Their review emphasized that the fractional order provides an additional parameter for adjusting the behaviour of image-processing operators.

Pu (2014) developed a fractional partial differential equation-based multiscale denoising model for texture-rich images. The study addressed a limitation of conventional PDE-based denoising: the potential loss of edge and texture details during smoothing. Its proposed framework used fractional calculus to improve the treatment of high-frequency structures.

Bai (2020) investigated image decomposition and denoising using fractional-order partial differential equations. The model separated an image into structural or cartoon-like components and texture-related components. Such decomposition is useful because image structure and fine texture may require different processing strategies.

RESEARCH METHODOLOGY

This paper adopts a literature-based analytical methodology. It draws on established publications in fractional calculus, mathematical image processing, fractional PDEs, and computer vision. The mathematical discussion is organized around derivative definitions, diffusion equations, spatial filtering, regularization, and image-quality evaluation.

The methodology consists of four stages:

Theoretical analysis: Examination of fractional derivatives and fractional Laplacian operators.

Model formulation: Construction of representative equations for denoising, enhancement, and restoration.

Application analysis: Interpretation of how fractional-order operators may support different image-processing tasks.

Comparative evaluation framework: Identification of mathematical and image-quality measures suitable for future experimental validation.

No new experimental dataset, measured performance values, or statistical test results are claimed in this paper. The proposed models are mathematical formulations that would require numerical implementation and empirical testing before any specific performance advantage could be established.

MATHEMATICAL FOUNDATIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

1. Concept of Fractional Calculus

Fractional calculus generalizes differentiation and integration from integer orders to arbitrary real or complex orders. In image processing, the fractional order provides a parameter for controlling how an operator responds to variations in image intensity and spatial frequency.

For a conventional first-order derivative,

$$\frac{df(x)}{dx},$$

the derivative measures the local rate of change of a function. In fractional calculus, the derivative can be written as

$$D^{\alpha}f(x), \quad \alpha > 0,$$

where α is the fractional order. For example, $\alpha = 1$ corresponds to the ordinary first derivative under the relevant definition, whereas a non-integer order such as $\alpha = 0.7$ represents a fractional derivative.

In a digital image, intensity is represented by a function

$$I(x, y) : \Omega \rightarrow \mathbb{R},$$

where Ω is the image domain and $(I(x, y))$ is the intensity at pixel coordinates (x, y) . Fractional-order spatial derivatives can be used to characterize variations in this intensity function.

The main attraction is the additional flexibility introduced by α , which allows the operator to be adjusted for different image characteristics (Yang et al., 2016).

2. Riemann–Liouville Fractional Derivative

For a function $(f(t))$, the left-sided Riemann–Liouville fractional derivative of order α , where $n - 1 < \alpha < n$ is defined as: -

$${}_a D_t^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau.$$

Here:

α is the derivative order.

$N = [\alpha]$ is the smallest integer greater than or equal to (α) .

$\Gamma(.)$ is the Gamma function.

a is the lower integration limit.

The integral involves values of the function over an interval rather than only its value at a single point. This nonlocal characteristic can be useful for modelling broader interactions in image intensity patterns. However, the derivative's exact interpretation depends on the selected definition, boundary conditions, and discretization.

3. Caputo Fractional Derivative

The Caputo fractional derivative of order α , for $n - 1 < \alpha < n$, is expressed as

$${}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n - \alpha - 1} f^{(n)}(\tau) d\tau.$$

The Caputo formulation differentiates the function before applying the fractional integral. It is often convenient in time-dependent mathematical models because initial conditions can be specified in terms of ordinary integer-order derivatives (Podlubny, 2012).

In image processing, time-fractional derivatives may describe the evolution of an image through successive denoising or enhancement iterations. They should be distinguished from spatial fractional derivatives, which act directly on the image coordinates.

4. Grunwald–Letnikov Approximation

For numerical implementation, the Grunwald–Letnikov formulation provides a finite-difference approximation to a fractional derivative. A common one-dimensional expression is

$$D^\alpha f(x) \approx \frac{1}{h^\alpha} \sum_{k=0}^N (-1)^k \binom{\alpha}{k} f(x - kh),$$

where h is the spatial sampling interval, N is the truncation length, and the generalized binomial coefficient is: -

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)}.$$

The coefficients depend on the fractional order. This approximation can be implemented using discrete image pixels and a fractional difference mask. In two-dimensional images, directional derivatives can be computed along horizontal, vertical, and diagonal directions, with appropriate treatment of image boundaries.

5. Fractional Laplacian

The fractional Laplacian is an important nonlocal operator used in fractional PDEs. For $0 < s < 1$, it can be defined on a suitable function u in $\mathbb{R}^d$ by

$$(-\Delta)^s u(x) = C_{d,s} \text{PV} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy,$$

where $C_{d,s}$ is a normalization constant and PV denotes the Cauchy principal value.

This definition indicates that the value at x is influenced by values at other locations y , with an interaction kernel depending on their distance. For suitable functions on an infinite or periodic domain, the Fourier representation is

$$\mathcal{F}\{(-\Delta)^s u\}(\xi) = |\xi|^{2s} \hat{u}(\xi),$$

where ξ denotes spatial frequency and $\hat{u}$ is the Fourier transform of u .

Equation provides a useful connection between fractional differential operators and frequency-domain image processing. The parameter s controls how the operator weights different frequencies. The resulting behaviour depends on whether the operator is used in a diffusion equation, a sharpening filter, or a regularization model (Bartels & Weber, 2023).

Applications of Fractional Differential Equations in Image Processing

1. Image Denoising

Image denoising is the process of removing unwanted disturbances while retaining useful image information. Noise may arise during image acquisition, transmission, compression, or sensor operation. Common examples include Gaussian noise, impulse noise, and signal-dependent noise.

A classical diffusion model represents image smoothing through the heat equation:

$$\frac{\partial I}{\partial t} = \Delta I,$$

where $I(x, y, t)$ represents the evolving image, t denotes processing time, and Δ is the spatial Laplacian.

A fractional-order spatial diffusion model can be written as

$$\frac{\partial I}{\partial t} = -\kappa(-\Delta)^s I, \quad 0 < s \leq 1,$$

where $\kappa > 0$ is the diffusion coefficient and s is the fractional order. The model is defined with suitable initial and boundary conditions.

In the Fourier domain, the solution to this constant-coefficient model has the form

$$\hat{I}(\xi, t) = e^{-\kappa t |\xi|^{2s}} \hat{I}_0(\xi),$$

where (I_0) is the initial noisy image.

Equation demonstrates that the diffusion process attenuates spatial frequencies at rates determined by s , κ , and t . However, the model is a linear smoother; it does not automatically preserve all edges. In practice, nonlinear diffusion, adaptive fractional orders, or regularization terms may be required to protect important structures.

Pu (2014) investigated fractional PDE-based multiscale denoising for texture images, while Bai (2020) explored fractional-order image decomposition and denoising. These studies provide a foundation for considering fractional operators in situations where noise reduction must be balanced against the preservation of texture.

Potential applications include:

Removing noise from medical scans.

Suppressing sensor noise in satellite images.

Improving low-light photographs.

Reducing disturbances in microscopy images.

Preprocessing images for segmentation and classification.

2. Edge Detection and Edge Preservation

Edges are locations where image intensity changes sharply. They are essential for identifying object boundaries, anatomical structures, roads, buildings, and other visual features.

A conventional gradient-based edge detector calculates

$$\nabla I = \left(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y} \right),$$

with gradient magnitude

$$G(x, y) = \sqrt{\left(\frac{\partial I}{\partial x} \right)^2 + \left(\frac{\partial I}{\partial y} \right)^2}.$$

A fractional gradient generalizes this formulation:

$$\nabla^\alpha I = (D_x^\alpha I, D_y^\alpha I),$$

where D_x^α and D_y^α denote fractional derivatives in the horizontal and vertical directions.

The associated fractional gradient magnitude is

$$G_\alpha(x, y) = \sqrt{|D_x^\alpha I|^2 + |D_y^\alpha I|^2}.$$

Edges can then be identified by thresholding $G\alpha$, although threshold selection and noise sensitivity must be considered carefully.

Fractional-order operators provide an additional degree of freedom for adjusting edge responses. They may help emphasize weak boundaries or textured structures that are not adequately represented by a particular integer-order filter. Nevertheless, the performance depends on the derivative definition, filter coefficients, image noise, and chosen threshold.

The computer-vision survey by Applications of Fractional Calculus in Computer Vision (2022) discusses the use of fractional derivatives in edge detection and related feature-extraction tasks.

3. Image Sharpening

Image sharpening increases the visibility of fine structures and transitions. Conventional sharpening methods often amplify high-frequency information, which can improve detail visibility but may also increase noise.

A fractional sharpening operator can be formulated as

$$I_{\text{sharp}} = I + \lambda D^{\alpha} I,$$

where $D^{\alpha} I$ denotes a suitably defined fractional derivative response and λ controls the sharpening strength.

For two-dimensional images, a Laplacian-based version is

$$I_{\text{sharp}} = I + \lambda (-\Delta)^s I.$$

The sign convention and implementation determine whether the operator acts as sharpening or smoothing. Under the convention in Equation, positive λ adds a high-frequency-weighted response and can enhance edges, but it can also amplify noise.

A practical implementation should therefore apply noise suppression before sharpening or use a regularized, edge-aware formulation. The sharpening strength may be adjusted according to local contrast and noise estimates.

Fractional sharpening may be useful for:

Enhancing fine structures in medical images.

Improving the visibility of roads and building boundaries in remote-sensing images.

Revealing surface textures in industrial inspection.

Improving detail visibility in photographs with moderate blur.

4. Contrast Enhancement

Contrast enhancement improves the distinction between intensity levels. It is particularly important when images contain poorly illuminated regions, weak boundaries, or a narrow intensity range.

A general fractional-gradient enhancement model can be written as: -

$$I_{\text{enh}}(x, y) = I(x, y) + \lambda(x, y) G_{\alpha}(x, y),$$

where $\lambda(x, y)$ is a spatially varying enhancement coefficient.

The coefficient can be designed to increase the visibility of weak structures while limiting amplification in highly noisy regions. For example,

$$\lambda(x, y) = \frac{\lambda_0}{1 + \beta \sigma_n^2(x, y)},$$

where $\lambda_0 > 0$ is the maximum enhancement strength, $\beta > 0$ controls noise sensitivity, and $\sigma_n^2(x, y)$ is a local estimate of noise variance.

Equation is a proposed adaptive design, not an experimentally validated optimal rule. It illustrates how local noise estimates could be used to reduce excessive enhancement in noisy areas.

Fractional enhancement can be combined with histogram equalization, adaptive contrast-limited histogram equalization, gamma correction, or illumination estimation. Its benefit should be evaluated against these established methods using both quantitative metrics and visual inspection.

Huang et al. (2024) examined fractional differential and integral operators for image enhancement and denoising, reporting potential benefits for weak-edge and texture enhancement.

II. CONCLUSION

Fractional differential equations offer a flexible mathematical framework for digital image processing and enhancement. By extending ordinary differentiation to non-integer orders, fractional calculus introduces additional modelling possibilities for image denoising, edge detection, sharpening, contrast enhancement, restoration, segmentation, and texture analysis.

The Riemann–Liouville and Caputo derivatives provide alternative definitions of fractional differentiation, while the Grunwald–Letnikov approximation supports numerical implementation on discrete image data. The fractional Laplacian offers a particularly useful connection between nonlocal spatial interactions and frequency-domain analysis. These mathematical tools can be integrated into diffusion equations and regularized optimization models to address the challenges of noise reduction and image-detail preservation.

The reviewed literature demonstrates substantial research interest in fractional-order methods. Nevertheless, their effectiveness depends on the selected operator, fractional order, numerical scheme, boundary conditions, and image characteristics. Linear fractional diffusion, for example, may suppress noise but can also attenuate important details. Similarly, fractional sharpening may enhance edges while amplifying noise. Adaptive modelling and rigorous performance evaluation are therefore essential.

In conclusion, fractional differential equations represent a promising extension of conventional image-processing mathematics, rather than a universal replacement for integer-order methods. Their practical value will depend on efficient algorithms, systematic parameter selection, and reproducible comparisons with established techniques. Further experimental research can clarify where fractional-order models provide the greatest benefits and how they can be incorporated into reliable image-processing systems.

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