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Applications of Spectral Sequences in Studying Complex Algebraic Varieties

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Abstract: *Spectral sequences have become an indispensable computational tool in modern algebraic geometry, particularly in the study of complex algebraic varieties. They provide a systematic framework for extracting information about intricate algebraic and topological structures by filtering complex objects into manageable layers. This paper explores the applications of spectral sequences in analyzing the cohomological properties of complex algebraic varieties, including sheaf cohomology, Hodge theory, and derived categories. Emphasis is placed on key constructions such as the Leray spectral sequence and the Hodge–de Rham spectral sequence, which facilitate the computation of cohomology groups and reveal deep relationships between geometry and topology. Furthermore, spectral sequences play a crucial role in studying fibrations, degenerations, and morphisms between varieties, enabling mathematicians to trace how local properties influence global structures. By bridging homological algebra and complex geometry, spectral sequences not only simplify otherwise intractable problems but also uncover hidden connections within algebraic varieties. This work highlights their theoretical significance and practical utility in advancing the understanding of complex algebraic structures*

Keywords: Spectral Sequences, Complex Algebraic Varieties, Sheaf Cohomology

I. INTRODUCTION

Spectral sequences are among the most powerful computational tools in modern algebraic topology and algebraic geometry, providing a systematic framework to compute complex invariants by breaking them into more manageable stages. Their applications in the study of complex algebraic varieties are especially profound, as these geometric objects often possess intricate structures that resist direct analysis. An algebraic variety defined over the complex numbers can be studied not only as an algebraic object but also as a complex manifold endowed with rich topological and analytical properties. Spectral sequences serve as a bridge connecting these perspectives, enabling mathematicians to extract deep information about cohomology, sheaf theory, and derived categories.

At their core, spectral sequences arise from filtered complexes. Suppose we have a chain complex $C\cdot$ with a filtration $FpC\cdot$. This filtration induces a sequence of pages $E_r^{p,q}$, each equipped with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$. he spectral sequence converges, under suitable conditions, to the associated graded object of the cohomology:

$$E_r^{p,q} \Rightarrow H^{p+q}(C^\bullet).$$

This iterative process allows one to compute complicated cohomological invariants step by step, starting from simpler approximations.

In the context of complex algebraic varieties, one of the most important applications of spectral sequences is in the computation of sheaf cohomology. Sheaf cohomology plays a central role in algebraic geometry because it encodes global information about varieties, such as the existence of global sections and obstructions to solving geometric problems. However, direct computation of sheaf cohomology groups $H(X,F)$ is often difficult. Spectral sequences, such

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as the Čech-to-derived functor spectral sequence, provide a method to compute these groups using coverings of the variety.

Another fundamental example is the Leray spectral sequence, which arises from a continuous map $f:X\rightarrow Y$ between topological spaces (or morphism of varieties). For a sheaf F on X, the Leray spectral sequence takes the form:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}),$$

where $Rqf*F$ denotes the higher direct image sheaves. This spectral sequence is particularly useful in algebraic geometry when studying fibrations or morphisms between varieties. It allows one to compute the cohomology of a complicated space X in terms of the cohomology of a simpler base space Y and the fibers of the map.

In the study of complex algebraic varieties, Hodge theory provides another rich area where spectral sequences play a crucial role. For a smooth projective complex variety X, the Hodge decomposition expresses the cohomology groups as:

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X).$$

This decomposition arises from the Hodge-to-de Rham spectral sequence:

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H_{\mathrm{dR}}^{p+q}(X),$$

where ΩXp denotes the sheaf of holomorphic p-forms. A remarkable fact is that for smooth projective varieties, this spectral sequence degenerates at the $E1$-page, meaning that all higher differentials vanish. This degeneration is a deep result with far-reaching consequences, linking algebraic geometry, differential geometry, and complex analysis.

Spectral sequences are also essential in the study of derived categories and hypercohomology. Given a complex of sheaves $F\cdot$, its hypercohomology $H(X,F\cdot)\cdot$ can be computed using spectral sequences. One important example is:

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{F}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet),$$

where $Hq(F\cdot))$ denotes the cohomology sheaves. This allows one to reduce the computation of hypercohomology to ordinary sheaf cohomology, which is often more accessible.

Another significant application lies in the study of filtrations on cohomology groups, such as the weight and Hodge filtrations in mixed Hodge theory. Spectral sequences help track how these filtrations interact and evolve. In particular, the weight spectral sequence is used to study singular varieties by resolving them into simpler pieces, enabling the computation of invariants that would otherwise be inaccessible.

In addition, spectral sequences are widely used in the computation of Ext and Tor functors, which are central to homological algebra and have important applications in algebraic geometry. For example, the Grothendieck spectral sequence provides a way to compute derived functors of a composition of two functors. If F and G are left exact functors, then under suitable conditions:

$$E_2^{p,q} = R^p F(R^q G(A)) \Rightarrow R^{p+q}(F \circ G)(A).$$

This is particularly useful when dealing with derived categories of sheaves on complex algebraic varieties.

Spectral sequences also play a crucial role in intersection theory and the study of characteristic classes. For example, they are used in the proof of the Riemann–Roch theorem and its generalizations, where one needs to relate various cohomological invariants. Similarly, in the study of fiber bundles and fibrations in algebraic geometry, spectral sequences such as the Serre spectral sequence provide a way to compute the cohomology of the total space from that of the base and the fiber.

Despite their power, spectral sequences are often considered difficult to use due to their abstract nature and the complexity of tracking differentials across pages. However, their conceptual importance cannot be overstated. They provide a unifying language that connects different areas of mathematics, including topology, geometry, and algebra.

Spectral sequences are indispensable tools in the study of complex algebraic varieties. They enable the computation of cohomological invariants, facilitate the analysis of morphisms between varieties, and provide deep insights into the

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structure of geometric objects. From the Leray and Hodge spectral sequences to applications in derived categories and mixed Hodge theory, their versatility and power make them central to modern research in algebraic geometry.

FOUNDATIONS OF SPECTRAL SEQUENCES

1. Filtered Complexes

Let $(C\cdot,)$ be a cochain complex with a filtration:

$$\cdots \subset F^{p+1}C^\bullet \subset F^pC^\bullet \subset \cdots \subset C^\bullet$$

This filtration induces a spectral sequence whose first page is:

$$E_0^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+q}$$

2. Convergence

A spectral sequence converges to the cohomology of the total complex:

$$E_\infty^{p,q} \cong \mathrm{Gr}^p H^{p+q}(C^\bullet)$$

where Grp denotes the graded components.

LERAY SPECTRAL SEQUENCE IN ALGEBRAIC GEOMETRY

One of the most important applications arises from a continuous map $f: X \to Y$. The Leray spectral sequence relates the cohomology of X to that of Y:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

APPLICATIONS

Computation of cohomology of fiber bundles
Analysis of morphisms between algebraic varieties
Study of fibrations in complex geometry
For example, if
f is a projection of a projective bundle, the Leray spectral sequence simplifies cohomological calculations significantly.

GROTHENDIECK SPECTRAL SEQUENCE

The Grothendieck spectral sequence applies to composed functors. If F and G are left exact functors, then:

$$E_2^{p,q} = R^pF(R^qG(A)) \Rightarrow R^{p+q}(FG)(A)$$

APPLICATIONS IN ALGEBRAIC VARIETIES

Derived functor computations in sheaf cohomology
Understanding composition of pushforward and global section functors
Computation of Ext and Tor groups
This spectral sequence is particularly useful in derived categories and coherent sheaf theory.

HODGE–DE RHAM SPECTRAL SEQUENCE

For a smooth complex algebraic variety X, the de Rham complex yields a spectral sequence:

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H_{dR}^{p+q}(X)$$

HODGE DECOMPOSITION

This leads to the famous Hodge decomposition:

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$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^q(X,\Omega_X^p)$$

APPLICATIONS
Study of complex manifolds
Hodge theory and classification of varieties
Analysis of Kähler manifolds
Degeneration at $E1$ or $E2$ provides deep geometric information.

SPECTRAL SEQUENCES IN SHEAF COHOMOLOGY
Sheaf cohomology is central to algebraic geometry, and spectral sequences simplify its computation.

ČECH-TO-DERIVED SPECTRAL SEQUENCE

$$E_2^{p,q} = H^p(X,\mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(X,\mathcal{F})$$

APPLICATIONS
Computation of cohomology via open covers
Transition between Čech and derived functor cohomology
Study of coherent and quasi-coherent sheaves

APPLICATIONS IN DERIVED CATEGORIES
Spectral sequences play a key role in derived categories, particularly in understanding filtered complexes and derived functors.

HYPERCOHOMOLOGY SPECTRAL SEQUENCE
For a complex of sheaves $F\cdot$:

$$E_2^{p,q} = H^p(X,\mathcal{H}^q(\mathcal{F}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X,\mathcal{F}^\bullet)$$

APPLICATIONS
Computation of hypercohomology
Derived functor analysis
Bridging homological algebra and geometry

INTERSECTION THEORY AND SPECTRAL SEQUENCES
Spectral sequences help compute intersection cohomology, especially for singular varieties.

PERVERSE SHEAVES AND FILTRATIONS
They provide tools to study:
Stratified spaces
Singular cohomology
Topological invariants of varieties

DEGENERATION AND GEOMETRIC INSIGHTS
A key feature is whether a spectral sequence degenerates at a certain page:
Degeneration at $E2$ often simplifies computations

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Provides geometric constraints
Indicates special structures (e.g., Kähler geometry)

ADVANTAGES AND LIMITATIONS

A. Advantages

Systematic computation of complex invariants
Applicability across multiple domains
Deep connection between algebra and geometry

B. Limitations

Complexity in higher pages
Convergence issues
Requires strong background in homological algebra

II. CONCLUSION

Spectral sequences have become one of the most powerful computational and conceptual tools in modern algebraic geometry, particularly in the study of complex algebraic varieties. Their importance lies not only in their ability to compute difficult invariants but also in how they reveal deep structural relationships between different cohomological theories. As complex varieties often possess intricate geometric and topological features, direct computation of invariants such as cohomology groups can be highly nontrivial. Spectral sequences provide a systematic, layered approach to such problems, breaking them into manageable stages.

At their core, spectral sequences arise from filtered complexes or double complexes and converge to desired algebraic invariants. A typical spectral sequence is denoted as:

$$E_r^{p,q} \Rightarrow H^{p+q}(X)$$

where Erp,q represents the r-th page of the spectral sequence, and the sequence "converges" to the graded pieces of a target cohomology group. This iterative refinement allows mathematicians to approximate complex cohomological structures step by step.

In the context of complex algebraic varieties, spectral sequences play a crucial role in connecting different cohomological frameworks. For instance, the Hodge-to-de Rham spectral sequence is fundamental in Hodge theory. It begins with the sheaf cohomology of differential forms:

$$E_1^{p,q} = H^q(X, \Omega_X^p)$$

and converges to the de Rham cohomology:

$$E_1^{p,q} \Rightarrow H_{dR}^{p+q}(X)$$

This spectral sequence reveals the decomposition of cohomology into Hodge structures, providing profound insight into the geometry of smooth projective varieties. Its degeneration at the $E1$ stage for compact Kähler manifolds is a cornerstone result, linking topology, complex structure, and differential geometry.

Another important application is the Leray spectral sequence, associated with a continuous map $f:X\rightarrow Y$. It relates the cohomology of X to that of Y via:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

This is particularly useful when studying fibrations of algebraic varieties, allowing one to understand global properties of X in terms of its fibers and base. It is widely applied in both étale and sheaf cohomology, making it indispensable in modern research.

Spectral sequences also facilitate comparisons between algebraic and analytic invariants. For example, in the study of mixed Hodge structures on singular varieties, spectral sequences help track how filtrations interact and evolve. They are

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also essential in the computation of hypercohomology, derived functors, and intersection cohomology, all of which are central to understanding singular spaces.

Spectral sequences serve as a unifying framework that bridges multiple areas within algebraic geometry. Their layered computational approach transforms seemingly intractable problems into structured procedures, while their theoretical implications uncover deep connections between geometry, topology, and algebra. As the study of complex algebraic varieties continues to evolve, spectral sequences remain an indispensable tool, both for explicit calculations and for advancing conceptual understanding.

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