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Algebraic Nature of Fuzzy Subgroups Under Homomorphism

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Abstract: *The study of fuzzy algebraic structures has significantly expanded the classical theory of groups by incorporating uncertainty and partial membership. In this context, fuzzy subgroups play a crucial role in bridging algebra with fuzzy set theory. This paper investigates the algebraic nature of fuzzy subgroups under group homomorphisms, focusing on the preservation and transformation of their structural properties. A fuzzy subgroup of a group is characterized by a membership function satisfying certain compatibility conditions with the group operation. When a homomorphism is applied between groups, it induces mappings on fuzzy subgroups, leading to the concepts of image and pre-image of fuzzy subgroups.*

The research examines how fundamental properties such as normality, level subsets, and support are affected under homomorphic mappings. It is shown that the pre-image of a fuzzy subgroup under a homomorphism is always a fuzzy subgroup, preserving essential algebraic characteristics. Furthermore, conditions under which the image of a fuzzy subgroup remains a fuzzy subgroup are analyzed, with particular emphasis on surjective homomorphisms. The study also explores the behavior of α-level cuts and demonstrates their consistency with classical subgroup structures under homomorphic transformations....

Keywords: Algebraic Structure, Fuzzy Algebra, Membership Function

I. INTRODUCTION

Zadeh was the first to introduce the fuzzy set. The concept of hazy subgroups was first introduced by Rosenfield. The investigation examined the impact of group homomorphism on fuzzy groups.

This chapter investigates the impact of group homomorphism on the level subgroups of fuzzy groups.

GROUP HOMOMORPHISM

If (G,.) and $(G^1,.)$ any two groups, then the function f is called a group homomorphism if.

$$f(xy) = f(x)f(y), \forall x, y \in G$$

Group Anti-Homomorphism

If (G,) and $(G^1,)$ any two groups, then the function f is called a group anti-homomorphism if.

$$f(xy) = f(y)f(x), \forall x, y \in G$$

Image of a fuzzy set and pre-image of a fuzzy set

is a fuzzy set and→Suppose S is a groupoid and f S I : is a fuzzy set→) (ϕ is a mapping and g S I : → : S S ϕ defined by

$$g(y) = \max_{x \in \phi^{-1}(y)} \{f(x)\}$$

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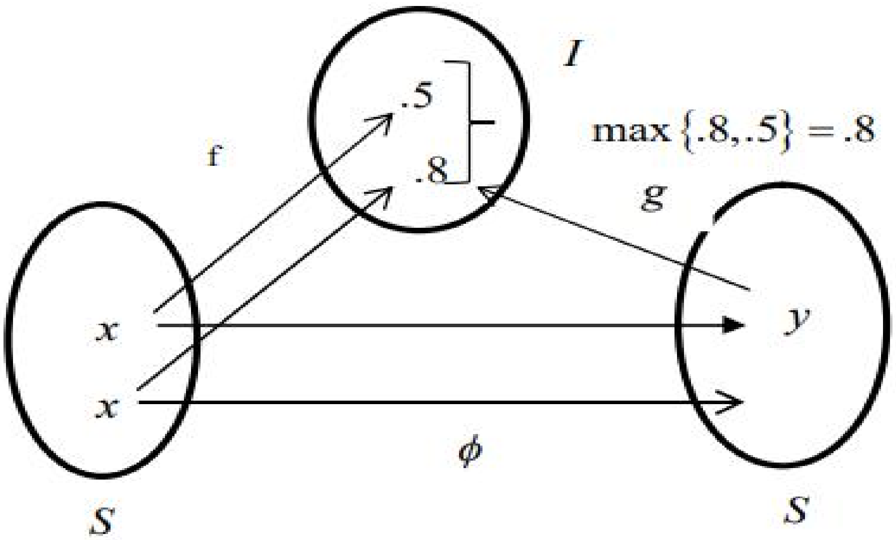

Figure 1: Image and Pre-Image of fuzzy set

SOME PROPOSITIONS

Proposition 1

The homomorphic image of a level subgroup of a fuzzy subgroup of a group G is a level subgroup of a fuzzy subgroup of a group G 1

Proof:

Let G and 1 G be any two groups. Let 1 be a homomorphism.→f G G : That is

$$f(xy) = f(x)f(y), \forall x, y \in G$$

Let, $V = f(A)$ where A is a fuzzy subgroup of a group G.

Clearly V is a fuzzy subgroup of a group G^1.

Let

$x, y \in G$

Implies $f(x)$ and $f(y)$ in G^1.

Clearly A_t is a level subgroup of A.

That is $A(x) \geq t$ and $A(y) \geq t$;

$A(xy^{-1}) \geq t$.

We have to prove that $f(A_t)$ is a level subgroup of V.

Now

$V(f(x)) \geq A(x) \geq t \Rightarrow V(f(x)) \geq t$;

$V(f(y)) \geq A(y) \geq t \Rightarrow V(f(y)) \geq t$

And $V\left(f(x)(f(y))^{-1}\right) = V\left(f(x)f(y^{-1})\right)$, as f is a

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homomorphism

$= V\left(f\left(xy^{-1}\right)\right)$, as f is a homomorphism.

$\geq A\left(xy^{-1}\right) \geq t$

Which implies that $V\left(f(x)(f(y))^{-1}\right) \geq t$

Hence $f(A_t)$ is a level subgroup of a fuzzy subgroup V of a group G^1.

Proposition 2

The homomorphic pre-image of a level subgroup of a fuzzy subgroup of a group 1 G is a level subgroup of a fuzzy subgroup of a group G.

Proof:

Let G and 1 G be any two groups. Let 1 be a homomorphism.→f G G :

Let G and G^1 be any two groups.

Let $f : G \rightarrow G^1$ be a homomorphism.

That is $f(xy) = f(x)f(y), \forall x, y \in G$

Let, $V = f(A)$ where V is a fuzzy subgroup of a group G^1.

Clearly A is a fuzzy subgroup of group G.

Let $f(x), f(y) \in G^1$, implies x and y in G.

Clearly $f(A_t)$ is a level subgroup of V.

That is $V(f(x)) \geq t$ and $V(f(y)) \geq t$;

$V\left(f(x)(f(y))^{-1}\right) \geq t$.

We have to prove that A_t is a level subgroup of A.

Now, $A(x) = V(f(x)) \geq t \Rightarrow A(x) \geq t$

$A(y) = V(f(y)) \geq t \Rightarrow A(y) \geq t$; And

$A\left(xy^{-1}\right) = V\left(f\left(xy^{-1}\right)\right)$

$= V\left(f(x)f\left(y^{-1}\right)\right)$, as f is a homomorphism

$= V\left(f(x)(f(y))^{-1}\right)$, as f is a homomorphism

$\geq t$,

Which implies that $A\left(xy^{-1}\right) \geq t$.

Hence A_t is a level subgroup of a fuzzy

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Proposition 3

The anti-homomorphic image of a level subgroup of a fuzzy subgroup of a group G is a level subgroup of a fuzzy subgroup of a group G 1.

Proof:

Let G and G 1 be any two groups. Let 1 be a anti-homomorphism.

That is $f(xy) = f(y)f(x), \forall x, y \in G$

Let, $V = f(A)$ where A is a fuzzy subgroup of a group G.

Clearly V is a fuzzy subgroup of a group G^1.

Let $x, y \in G$, implies $f(x)$ and $f(y)$ in G^1.

Clearly A_t is a level subgroup of A.

That is $A(x) \geq t$ and $A(y) \geq t$;

$A(y^{-1}x) \geq t$

We have to prove that $f(A_t)$ is a level subgroup of V.

Now $V(f(x)) \geq A(x) \geq t \Rightarrow V(f(x)) \geq t$;

$V(f(y)) \geq A(y) \geq t \Rightarrow V(f(y)) \geq t$

And $V\left(f(x)(f(y))^{-1}\right) = V(f(x)f(y^{-1}))$, as f is an anti-homomorphism

$= V(f(y^{-1}x))$ as f is an anti-homomorphism

$\geq A(y^{-1}x) \geq t$

Which implies that

$V\left(f(x)(f(y))^{-1}\right) \geq t$

Hence $f(A_t)$ is a level subgroup of a fuzzy subgroup V of a group G^1.

Proposition 4

The anti-homomorphism pre-image of a level subgroup of a fuzzy subgroup of a group G^1 is a level subgroup of a fuzzy subgroup of a group G.

Proof:

Let G and G^1 be any two groups.

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Let $f : G \rightarrow G^1$ be a anti-homomorphism.

That is

$f(xy) = f(y)f(x), \forall x, y \in G$

Let,

$V = f(A)$ Where V is a fuzzy subgroup of a group G^1

Clearly A is a fuzzy subgroup of group G

Let $f(x), f(y) \in G^1$, implies x and y in G.

Clearly $f(A_t)$ is a level subgroup of V.

That is $V(f(x)) \geq t$ and $V(f(y)) \geq t$

$V\left((f(y))^{-1} f(x)\right) \geq t.$

We have to prove that A_t is a level subgroup of A.

Now, $A(x) = V(f(x)) \geq t \Rightarrow A(x) \geq t$

$A(y) = V(f(y)) \geq t \Rightarrow A(y) \geq t$; And

$A(xy^{-1}) = V(f(xy^{-1}))$

$= V(f(y^{-1})f(x))$, as f is a anti-homomorphism

$= V\left((f(y))^{-1} f(x)\right) \geq t$, as f is a anti-homomorphism

Which implies that $A(xy^{-1}) \geq t$.

Hence A_t is a level subgroup of a fuzzy subgroup A of a group G.

II. CONCLUSION

The algebraic nature of fuzzy subgroups under homomorphism reveals a rich and coherent extension of classical group theory into the domain of fuzzy set theory, preserving structural properties while accommodating graded membership. A key conclusion is that homomorphisms play a fundamental role in maintaining the integrity of fuzzy subgroups across group mappings. When a fuzzy subgroup is mapped from one group to another via a homomorphism, its image and pre-image retain essential subgroup characteristics under appropriate conditions. In particular, the pre-image of a fuzzy subgroup under a group homomorphism is always a fuzzy subgroup, which demonstrates the stability of fuzzy algebraic structures under inverse mappings. This property ensures that the structural behavior of fuzzy subgroups is consistent with classical subgroup theory, where inverse images preserve subgroup properties.

Moreover, the image of a fuzzy subgroup under a homomorphism also forms a fuzzy subgroup, provided certain conditions such as normality or compatibility of membership functions are satisfied. This highlights how homomorphic images can preserve fuzziness while reflecting the algebraic transformation imposed by the mapping. The preservation of operations such as fuzzy intersection, union, and level subsets (α-cuts) under homomorphisms further emphasizes the compatibility between fuzzy set operations and algebraic structures. These α-cuts, which correspond to crisp subgroups at different levels of membership, behave predictably under homomorphic mappings, reinforcing the idea that fuzzy subgroups can be studied through their crisp counterparts.

Another significant aspect is the behavior of normal fuzzy subgroups under homomorphisms. The image of a normal fuzzy subgroup under a surjective homomorphism remains normal, which parallels classical results in group theory and supports the extension of quotient structures into the fuzzy context. Consequently, the concept of quotient groups can be generalized to fuzzy quotient groups, allowing for deeper exploration of algebraic structures where uncertainty or partial membership is inherent. This has important implications in areas such as decision-making systems, pattern recognition, and control theory, where fuzzy logic provides a more realistic representation of complex systems.

Furthermore, homomorphisms facilitate the classification and comparison of fuzzy subgroups by enabling equivalence relations and isomorphism-like conditions. These mappings help identify when two fuzzy subgroups share similar algebraic behavior, even if they belong to different groups. As a result, homomorphic relationships contribute to a broader understanding of the internal symmetry and structure of fuzzy algebraic systems.

The study of fuzzy subgroups under homomorphisms demonstrates that many foundational principles of classical algebra extend naturally into the fuzzy domain, with appropriate modifications to account for graded membership. Homomorphisms not only preserve the essential structure of fuzzy subgroups but also provide powerful tools for analyzing their properties and relationships. This interplay between fuzziness and algebraic structure underscores the robustness and versatility of fuzzy group theory, making it a valuable framework for both theoretical investigation and practical applications in uncertain and complex environments.

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