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Coupled Fixed Point Theorem with EA Property in Neutrosophic Metric Spaces

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Abstract: *In this paper, we establish a coupled fixed point theorem under the Existence and Approximation (EA) property in a neutrosophic met- ric space. The proposed framework generalizes the Banach contraction principle and extends classical fuzzy and intuitionistic fuzzy results by incorporating indeterminacy through neutrosophic components. The EA property eliminates the need for continuity assumptions while en- suring convergence and uniqueness of the coupled fixed point. An illus- trative example demonstrates the validity of the obtained results.*

Keywords: Coupled fixed point theorem, Neutrosophic metric space, EA property, Compatibility, Indeterminacy, Nonlinear analysis

Mathematics Subject Classification: 54H25, 47H10

I. INTRODUCTION

In this section, we recall some fundamental notions and definitions that are essential for the development of our main results. The evolution of these concepts spans from fuzzy sets introduced by Zadeh [1] to neutrosophic metric spaces proposed by Smarandache [5]. Intermediate generalizations include fuzzy metric spaces by Kramosil and Michalek [2], intuitionistic fuzzy sets by Atanassov [4], and their metric extensions by George and Veeramani [3]. Recent studies by Kirisi and Simsek [6] and Jeyaraman and Sowndrarajan [7] extended these ideas to neutrosophic contexts.

Definition 1.1 A fuzzy metric space is a triple $(X, M, *)$, where X is a nonempty set, $M : X \times X \times (0, \infty) \to [0, 1]$ satisfies:

$$M(x, y, t) = 1 \iff x = y,\ M(x, y, t) = M(y, x, t),$$

and for all $x, y, z \in X$, $s, t > 0$,

$$M(x, z, t + s) \geqslant M(x, y, t) * M(y, z, s),$$

where $*$ is a continuous t-norm. This was the first extension of the classical metric concept to fuzzy environments.

Definition 1.2 An intuitionistic fuzzy set A in a universe X is character- ized by a membership function $\mu A : X \to [0, 1]$ and a non-membership function $\nu A : X \to [0, 1]$, satisfying

$$0 \leqslant \mu A(x) + \nu A(x) \leqslant 1.$$

This concept generalizes Zadehs fuzzy sets by accounting for both the degree of membership and non-membership.

Definition 1.3 A neutrosophic set A on X is defined by three independent functions:
$TA(x), IA(x), FA(x) : X \to [0, 1]$,
where TA(x), IA(x), and FA(x) denote the degrees of truth, indeterminacy, and falsity, respectively, without the restriction $TA + IA + FA = 1$. This structure allows representation of incomplete, inconsistent, and indeterminate informa- tion simultaneously.

Definition 1.4 (Kirisi and Simsek [6]) A neutrosophic metric space (NMS) is a 5-tuple (Ξ, M, N, O, $*$, $\diamond$), where

- Ξ is a nonempty set,
- M, N, O : $\Xi \times \Xi \times (0, \infty) \to [0, 1]$ are mappings representing nearness, non-nearness, and indeterminacy,
- $*$, $\diamond$ are continuous t-norm and t-conorm, respectively, and for all $x, y, z \in \Xi$, $t, s > 0$:

1. $M(x, y, t) + N(x, y, t) + O(x, y, t) \leqslant 3$,
2. $M(x, y, t) = 1 \Longleftrightarrow x = y$,
3. $M(x, z, t + s) \geqslant M(x, y, t) * M(y, z, s)$,
4. $N(x, z, t + s) \leqslant N(x, y, t) \diamond N(y, z, s)$,
5. $O(x, z, t + s) \leqslant O(x, y, t) \diamond O(y, z, s)$,
6. M, N, O are continuous in all parameters.

Note: This generalizes both fuzzy and intuitionistic fuzzy metrics by integrating indeterminacy (O).

Definition 1.5 A sequence $\{x_n\}$ in Ξ is said to converge to $x \in \Xi$ if for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \lim_{n\to\infty} N(x_n, x, t) = 0, \quad \lim_{n\to\infty} O(x_n, x, t) = 0.$$

This definition follows the standard convergence in fuzzy metrics [3], extended to neutrosophic structures [6].

Definition 1.6 A sequence $\{x_n\}$ in Ξ is called Cauchy if for every $\epsilon > 0$ and $t > 0$, there exists $n_0 \in N$ such that for all $m, n \geqslant n_0$:

$M(x_m, x_n, t) > 1 - \epsilon$, $N(x_m, x_n, t) < \epsilon$, $O(x_m, x_n, t) < \epsilon$.

Definition 1.7 (Kramosil & Michalek [2]) A neutrosophic metric space (Ξ, M, N, O, $*$, $\diamond$) is said to be complete if every Cauchy sequence in Ξ con- verges to a point in Ξ. This extends completeness from classical and fuzzy metric spaces to the neutrosophic setting.

Definition 1.8 (Bhaskar and Lakshmikantham) Let $F, G : \Xi \times \Xi \to \Xi$. A pair $(x, y) \in \Xi \times \Xi$ is called a coupled fixed point of (F, G) if

$F(x, y) = x$, $G(y, x) = y$.

The notion of coupled fixed point was initially introduced for partially ordered metric spaces and has been extended to fuzzy and neutrosophic settings [6, 7].

Definition 1.9 (Ishtiaq et al. [9]) A pair of mappings (F, G) satisfies the EA property if there exist sequences $\{x_n\}$, $\{y_n\} \subset \Xi$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} G(y_n, x_n) = z,$$

for some $z \in \Xi$. This property allows the existence of coincidence points without requiring continuity, and serves as a foundation for fixed point results in neutrosophic spaces.

Definition 1.10 (Jeyaraman and Sowndrarajan [7]) Mappings $F, G : \Xi \times \Xi \to \Xi$ are said to commute at their coincidence points if

$F(x, y) = G(x, y) \Rightarrow F(G(x, y), G(y, x)) = G(F(x, y), F(y, x))$.

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II. MAIN THEOREM

Theorem 2.1 Let (Ξ, M, N, O, ∗,) be a complete neutrosophic metric space, and let F, G : Ξ × Ξ → Ξ be two self-mappings satisfying:

(E1) There exist sequences {xn}, {yn} ⊂ Ξ such that

$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} G(y_n, x_n) = z$, (1)

for some z ∈ Ξ.

(E2) The range G(Ξ × Ξ) is closed in Ξ.

(E3) There exists k ∈ (0, 1) such that for all x, y, u, v ∈ Ξ and t > 0,

M (F (x, y), F (u, v), t) ⩾ M (G(x, y), G(u, v), kt),

N (F (x, y), F (u, v), t) ⩽ N (G(x, y), G(u, v), kt),

O(F (x, y), F (u, v), t) ⩽ O(G(x, y), G(u, v), kt).

(E4) F and G commute at their coincidence points.

Then there exists a unique coupled fixed point (x∗, y∗) ∈ Ξ × Ξ such that

F (x∗, y∗) = x∗, G(y∗, x∗) = y∗.

Proof

Take arbitrary (x0, y0) ∈ Ξ and define:

$x_{n+1} = F(x_n, y_n)$, $y_{n+1} = G(y_n, x_n)$, n ⩾ 0. (2) This gives two sequences {xn} and {yn} in Ξ.

Using the contraction condition, contractM, for all t > 0:

M (xn+1, xn+2, t) = M (F (xn, yn), F (xn+1, yn+1), t) ⩾ M (G(xn, yn), G(xn+1, yn+1), kt).

(3)

Similarly,

$$M(y_{n+1}, y_{n+2}, t) \geqslant M(G(y_n, x_n), G(y_{n+1}, x_{n+1}), kt). \quad (4)$$

By induction, we get:

$$M(x_n, x_{n+p}, t) \geqslant M(x_0, x_1, k^p t), \quad (5)$$

and analogously for {yn}.

Since 0 < k < 1, as p → ∞, kpt → 0, and from neutrosophic metric properties:

$\lim_{m,n\to\infty} M(x_m, x_n, t) = 1$, $\lim_{m,n\to\infty} N(x_m, x_n, t) = 0$, $\lim_{m,n\to\infty} O(x_m, x_n, t) = 0$.

Thus, {xn} and {yn} are Cauchy sequences.

Since (Ξ, M, N, O, ∗,) is complete, every Cauchy sequence converges. Hence, there exist x∗, y∗ ∈ Ξ such that

xn → x∗, yn → y∗, (6)

i.e.,

$\lim_{n\to\infty} M(x_n, x_*, t) = 1$, $N(x_n, x_*, t) = O(x_n, x_*, t) = 0$.

From (E1) and (E2), there exist {xn}, {yn} satisfying EA property such that their common limit z = lim F (xn, yn) = lim G(yn, xn) belongs to G(Ξ × Ξ). Hence, there exists (u, v) ∈ Ξ × Ξ such that:

z = G(u, v). (7)

By the continuity of M, N, O and contract M , one obtains:

M (F (u, v), G(u, v), t) = 1, N (F (u, v), G(u, v), t) = O(F (u, v), G(u, v), t) = 0,

which implies

F (u, v) = G(u, v). (8)

Taking limits in the iterative definitions sequence and using limit, we have:

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$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x_*, y_*),$$

and

$$\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} G(y_n, x_n) = G(y_*, x_*).$$

But by convergence of $\{x_n\}$, $\{y_n\}$:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x_*, \quad \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} y_n = y_*.$$

Therefore,

$F(x_*, y_*) = x_*$, $G(y_*, x_*) = y_*$. (9)

Hence (x_*, y_*) is a coupled fixed point of (F, G).

Let (x', y') be another coupled fixed point, i.e.,

$F(x', y') = x'$, $G(y', x') = y'$. (10)

Using contract M and fixed point, fixed2, for any $t > 0$:

$M(x_*, x', t) = M(F(x_*, y_*), F(x', y'), t) \geqslant M(G(x_*, y_*), G(x', y'), kt) = M(y_*, y', kt)$.

Similarly,

$M(y_*, y', t) \geqslant M(x_*, x', kt)$.

Combining, we have:

$M(x_*, x', t) \geqslant M(x_*, x', k^{2n}t)$. (11)

As $n \to \infty$, $k^{2n}t \to 0$, hence $M(x_*, x', t) = 1$ and $M(y_*, y', t) = 1$. Thus $x_* = x'$ and $y_* = y'$.

By EA property, contraction, and completeness, the pair (x_*, y_*) in fixed point is the unique coupled fixed point in $(\Xi, M, N, O, *, \diamond)$.

Example 2.2 Let $\Xi = \mathbb{R}$ with neutrosophic metric components:

$$M(x, y, t) = \frac{t}{t + |x - y|}, \quad N(x, y, t) = \frac{|x - y|}{t + |x - y|}, \quad O(x, y, t) = \frac{|x - y|}{t + 1},$$

and continuous t-norm, t-conorm: $a * b = \min(a, b)$, $a \diamond b = \max(a, b)$.

Define mappings $F, G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *as:*

$$F(x, y) = \frac{x + y}{6}, \quad G(x, y) = \frac{x + y}{8}.$$

(i) *For any sequence* $x_n = y_n = \frac{1}{n}$,

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} \frac{2/n}{6} = 0, \quad \lim_{n\to\infty} G(y_n, x_n) = \lim_{n\to\infty} \frac{2/n}{8} = 0.$$

Thus the EA property holds with $z = 0$.

(ii) $G(\mathbb{R} \times \mathbb{R}) = \mathbb{R}$, *hence closed.*

(iii) *For* $k = \frac{3}{4}$ *and any* $x, y, u, v \in \mathbb{R}$,

$$M(F(x, y), F(u, v), t) = \frac{t}{t + \frac{|x+y-u-v|}{6}} \geq \frac{t}{t + \frac{3|x+y-u-v|}{8}} = M(G(x, y), G(u, v), kt).$$

Hence the contraction condition holds.

(iv) *F and G commute since*

$$F(G(x, y), G(y, x)) = G(F(x, y), F(y, x)) = \frac{x + y}{12}.$$

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By the theorem, $(x*, y*) = (0, 0)$ is the unique coupled fixed point since:
$F(0, 0) = G(0, 0) = 0$.
This example satisfies all hypotheses of the theorem. The EA property ensures existence of the sequence approaching (0, 0), contraction ensures the Cauchy property, completeness implies convergence, and (0, 0) is the unique coupled fixed point.

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