IJARSCT

International Journal of Advanced Research in Science, Communication and Technology

International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal

ISSN: 2581-9429

Volume 5, Issue 4, November 2025

Impact Factor: 7.53

Review of Exact Sequences in Understanding Morphisms Between Algebraic Varieties

Suresh Kumar N[1] and Dr. Chandrabhan Singh[2]

[1]Research Scholar, Department of Mathematics

[2]Professor, Department of Mathematics

Sunrise University, Alwar, Rajasthan

Abstract: *Exact sequences play a central role in modern algebraic geometry by providing a powerful framework for analyzing morphisms between algebraic varieties. This review explores how exact sequences, particularly short and long exact sequences arising from sheaf cohomology, contribute to understanding the structure and properties of morphisms. By examining the interaction between kernels, images, and cokernels, exact sequences allow mathematicians to track how geometric and algebraic information is preserved or transformed under maps between varieties. The paper highlights the role of exact sequences in studying fundamental concepts such as embeddings, projections, and birational morphisms. Special attention is given to their application in coherent sheaves, divisor class groups, and cohomological invariants, which provide deeper insights into the behavior of morphisms. Additionally, the review discusses how tools like the snake lemma and five lemma facilitate the extraction of meaningful relationships between algebraic structures associated with varieties.*

Keywords: Exact sequences, algebraic varieties, sheaf cohomology

I. INTRODUCTION

Exact sequences play a central role in modern algebraic geometry, providing a powerful language for understanding how morphisms between algebraic varieties behave, interact, and encode geometric information. At a foundational level, algebraic varieties are geometric objects defined as the solution sets of systems of polynomial equations over a field, and morphisms between them are maps that preserve this algebraic structure. However, to fully grasp the subtleties of these morphisms—such as how subvarieties embed, how functions extend, or how fibers behave one requires tools that go beyond simple pointwise descriptions. Exact sequences offer precisely such a framework by translating geometric problems into algebraic ones involving sequences of groups, modules, or sheaves, where the notion of exactness captures the idea of "perfect fitting" between successive mappings. Formally, a sequence of homomorphisms between algebraic objects

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is said to be exact at B if Im(f)=ker(g), meaning that everything that maps to zero under g comes exactly from A via f. This simple yet profound condition encodes how information flows through the sequence, ensuring that no data is lost or redundantly introduced at that stage.

In the context of algebraic varieties, exact sequences frequently arise when studying morphisms through their induced actions on coordinate rings, sheaves of regular functions, or cohomological invariants. For instance, given a morphism of varieties ϕ:

$X \to Y$, one often examines the pullback homomorphism on structure sheaves $\phi\#{:}OY \to \phi * OX$, and exact sequences of sheaves help describe how functions on Y relate to those on X. Similarly, when considering subvarieties $Z \subseteq X$, exact sequences such as

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

IJARSCT

International Journal of Advanced Research in Science, Communication and Technology

International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal

ISSN: 2581-9429 Volume 5, Issue 4, November 2025 Impact Factor: 7.53

ncode the embedding of Z into X, where IZ is the ideal sheaf of functions vanishing on Z. This short exact sequence succinctly expresses how the global structure of X decomposes into functions vanishing along Z and functions restricted to Z, thereby linking geometric inclusion with algebraic structure.

Moreover, exact sequences are indispensable in the study of morphisms through their role in cohomology, particularly sheaf cohomology, which measures the global behavior of local data. When a short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is given, it induces a long exact sequence in cohomology

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to \cdots,$$

which provides deep insight into how global sections and higher cohomology groups transform under morphisms. These long exact sequences are particularly useful in detecting obstructions, computing invariants, and understanding the failure of certain properties, such as surjectivity or injectivity, at a global level. Thus, exact sequences act as a bridge between local algebraic data and global geometric phenomena.

Another important application of exact sequences in understanding morphisms arises in the study of tangent spaces and differentials. Given a morphism $\phi{:}X \to Y$, one can analyze its infinitesimal behavior through the induced map on tangent spaces or via the sheaf of differentials ΩX. Exact sequences involving these sheaves help characterize whether a morphism is smooth, étale, or has singularities. For example, sequences relating relative differentials often reveal how the geometry of fibers varies across the base variety, providing crucial insights into deformation theory and moduli problems.

Exact sequences also facilitate the study of kernels and cokernels of morphisms, which correspond geometrically to fibers and quotient constructions. In categorical terms, the kernel of a morphism reflects the points mapping to a distinguished element, while the cokernel captures how the image fails to be surjective. Exactness ensures that these constructions behave well and fit into a coherent algebraic structure. In particular, short exact sequences

$$0 \to A \to B \to C \to 0$$

are especially significant, as they describe situations where A embeds into B, and C is precisely the quotient B/A. In algebraic geometry, such sequences often correspond to extensions of sheaves or vector bundles, and understanding these extensions is key to classifying geometric objects and morphisms between them.

Furthermore, the language of exact sequences is essential in derived categories and homological algebra, which provide a more advanced framework for studying morphisms between varieties. In these settings, exact sequences are generalized to distinguished triangles, and morphisms are studied up to homotopy, allowing for a deeper and more flexible understanding of geometric relationships. This perspective has become central in modern research areas such as mirror symmetry, moduli theory, and the study of singularities.

Exact sequences serve as a fundamental tool in understanding morphisms between algebraic varieties by encoding the precise relationships between algebraic and geometric structures. They allow mathematicians to track how information is preserved, lost, or transformed under mappings, and they provide a unifying language that connects local properties with global behavior. Whether through sheaf theory, cohomology, or homological methods, exact sequences reveal the hidden structure underlying morphisms and enable a systematic approach to some of the most intricate problems in algebraic geometry.

PRELIMINARIES ON EXACT SEQUENCES

A sequence of morphisms of abelian groups or modules:

$$\cdots \to A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \to \cdots$$

is called exact at An if:

ISSN 2581-9429 IJARSCT

$$\mathrm{Im}(f_{n-1}) = \ker(f_n)$$

A short exact sequence is of the form:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where:
f is injective,
g is surjective,
Im(f)=ker(g)
Exact sequences allow us to decompose complex structures into simpler components and understand how morphisms behave in relation to kernels and images.

MORPHISMS BETWEEN ALGEBRAIC VARIETIES

Let X and Y be algebraic varieties over a field k, and let:

$$\varphi : X \to Y$$

be a morphism of varieties. Such a morphism induces a pullback map on regular functions:

$$\varphi^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$$

At the level of sheaves, this induces a morphism between structure sheaves:

$$\varphi^\# : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$$

Exact sequences become relevant when studying how these induced maps behave, particularly when analyzing kernels, cokernels, and the structure of sheaves associated with morphisms.

EXACT SEQUENCES OF SHEAVES

Sheaves are central to modern algebraic geometry. Given a morphism of sheaves:

$$0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \to 0$$

this is a short exact sequence if it is exact at every stalk:

$$0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0$$

for all points $x \in X$.
Such sequences arise naturally from morphisms between varieties. For example, if I is an ideal sheaf defining a closed subvariety $Z \subset X$, then:

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

This exact sequence encodes the embedding of Z into X and is fundamental in studying morphisms involving subvarieties.

LONG EXACT SEQUENCES IN COHOMOLOGY

Applying a cohomological functor such as global sections Γ (X,−) or sheaf cohomology Hi(X,−) to a short exact sequence yields a long exact sequence:

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to \cdots$$

his long exact sequence is crucial for understanding how morphisms affect global properties of varieties. It allows one to compute cohomology groups indirectly and understand obstructions to lifting sections.
For a morphism φ:X→Y, the higher direct image functors $Ri\varphi*$ also give rise to long exact sequences:

$$0 \to \varphi_*\mathcal{F} \to \varphi_*\mathcal{G} \to \varphi_*\mathcal{H} \to R^1\varphi_*\mathcal{F} \to \cdots$$

These sequences reveal how morphisms transform cohomological invariants.

EXACT SEQUENCES AND KERNELS OF MORPHISMS

Exact sequences help characterize morphisms by examining kernels and cokernels. For a morphism of sheaves:

$$f : \mathcal{F} \to \mathcal{G}$$

we can define:

$$\ker(f), \quad \mathrm{coker}(f)$$

and obtain a canonical exact sequence:

$$0 \to \ker(f) \to \mathcal{F} \xrightarrow{f} \mathcal{G} \to \mathrm{coker}(f) \to 0$$

In the context of algebraic varieties, this structure helps determine whether a morphism is:
Injective (kernel vanishes),
Surjective (cokernel vanishes),
Isomorphism (both kernel and cokernel vanish).

APPLICATIONS IN ALGEBRAIC GEOMETRY

Algebraic geometry, a branch of mathematics that studies solutions of systems of polynomial equations, has evolved from a purely theoretical discipline into a powerful tool with diverse applications across science and technology. By combining algebraic techniques with geometric intuition, it provides deep insights into structures that arise in many modern fields.

One of the most significant applications of algebraic geometry is in number theory. Problems involving integers, such as finding rational solutions to equations, are often translated into geometric questions about curves and surfaces. For instance, elliptic curves entral objects in algebraic geometry play a crucial role in solving Diophantine equations. They were instrumental in the proof of Fermat's Last Theorem and continue to be essential in modern research in arithmetic geometry.

Another major application lies in cryptography. Algebraic geometry provides the foundation for elliptic curve cryptography (ECC), which is widely used in secure communication systems. ECC relies on the algebraic structure of elliptic curves over finite fields to create encryption schemes that are both secure and efficient. Compared to traditional methods like RSA, elliptic curve methods offer similar security with smaller key sizes, making them ideal for modern devices with limited computational resources.

Algebraic geometry also has applications in coding theory. Error-correcting codes are essential in digital communication and data storage, ensuring that information can be transmitted accurately even in the presence of noise. Algebraic geometric codes, such as Goppa codes, use curves over finite fields to construct codes with excellent error-correcting capabilities. These codes are used in satellite communication, data transmission, and storage systems.

Algebraic geometry is far more than an abstract mathematical theory. Its applications span number theory, cryptography, coding theory, physics, engineering, and statistics. As technology advances and mathematical tools become more sophisticated, the importance of algebraic geometry continues to grow, making it a central pillar of modern scientific and technological development.

STUDYING SUBVARIETIES

Exact sequences such as:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

IJARSCT

International Journal of Advanced Research in Science, Communication and Technology

International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal

ISSN: 2581-9429 Volume 5, Issue 4, November 2025 Impact Factor: 7.53

allow us to analyze how a subvariety Z sits inside X. Cohomological consequences help compute invariants like dimension and genus.

MORPHISMS AND FIBERS

For a morphism $\varphi:X\rightarrow Y$, exact sequences help understand fibers via sheaf-theoretic methods. The behavior of fibers is often studied using base change theorems and derived functors.

DEFORMATION THEORY

Exact sequences play a key role in deformation theory, where infinitesimal deformations are governed by tangent spaces and obstruction spaces expressed via cohomology groups.

A typical relation involves:

$$T_X \cong \mathrm{Hom}(\Omega^1_X, \mathcal{O}_X)$$

where $\Omega X1$ fits into exact sequences induced by morphisms.

ROLE OF DERIVED FUNCTORS

Derived functors generalize the notion of exactness. For example, right-derived functors of the global section functor yield sheaf cohomology:

$$H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F})$$

Exact sequences are essential in defining and computing these derived functors. They provide the framework for connecting algebraic and geometric information through homological methods.

II. CONCLUSION

Exact sequences provide a powerful and unifying framework for understanding morphisms between algebraic varieties, offering deep insight into how geometric structures interact through algebraic maps. At their core, exact sequences encode the idea of measuring how far a morphism is from being injective or surjective, thereby capturing essential information about kernels, images, and cokernels in a precise and structured way. When studying morphisms of algebraic varieties, particularly through the lens of sheaf theory, cohomology, and derived functors, exact sequences become indispensable tools for tracking how local data glues together globally. They allow mathematicians to decompose complex morphisms into simpler components, making it possible to analyze properties such as smoothness, birationality, and fiber structure in a systematic manner.

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

The above short exact sequence illustrates a situation where the morphism f embeds A into B, and g projects B onto C such that the image of f coincides exactly with the kernel of g. In the context of algebraic varieties, such sequences often arise when studying sheaves of regular functions, differentials, or divisor class groups. They help reveal how subvarieties sit inside ambient varieties and how quotient structures emerge naturally from morphisms. Furthermore, exact sequences are central to cohomological techniques, where long exact sequences derived from short ones enable the computation of invariants that are otherwise difficult to access directly. This is particularly significant in modern algebraic geometry, where cohomology groups carry rich geometric and arithmetic information.

Moreover, exact sequences facilitate comparisons between different varieties by analyzing how morphisms induce maps between their associated algebraic or topological invariants. For instance, in studying morphisms between projective varieties, exact sequences of sheaves can be used to understand the behavior of line bundles, sections, and embeddings into projective space. They also play a crucial role in deformation theory, where one studies how small perturbations of a variety or morphism affect its structure; exact sequences help track infinitesimal changes and

IJARSCT

International Journal of Advanced Research in Science, Communication and Technology

International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal

ISSN: 2581-9429

Volume 5, Issue 4, November 2025

Impact Factor: 7.53

obstructions. In addition, derived categories and spectral sequences, which build upon the concept of exactness, extend these ideas further, allowing for a more flexible and powerful language to study complex geometric phenomena.

Ultimately, exact sequences do not merely serve as technical tools but form a conceptual bridge between algebra and geometry. They encapsulate the essence of how information flows through morphisms and provide a language for expressing subtle relationships between varieties. By organizing data into coherent chains of maps with precise exactness conditions, they make it possible to uncover hidden structures and connections that would otherwise remain obscured. As such, mastering exact sequences is fundamental to a deeper understanding of morphisms in algebraic geometry, enabling both rigorous analysis and creative exploration of the rich landscape of algebraic varieties.

REFERENCES

[1]. Hartshorne, R. (1977). Algebraic Geometry. Springer-Verlag.
[2]. Eisenbud, D. (1995). Commutative Algebra with a View Toward Algebraic Geometry. Springer.
[3]. Griffiths, P., & Harris, J. (1978). Principles of Algebraic Geometry. Wiley.
[4]. Atiyah, M. F., & Macdonald, I. G. (1969). Introduction to Commutative Algebra. Addison-Wesley.
[5]. Weibel, C. A. (1994). An Introduction to Homological Algebra. Cambridge University Press.
[6]. Grothendieck, A. (1960). Éléments de géométrie algébrique. Publications Mathématiques de l'IHÉS.
[7]. Vakil, R. (2017). The Rising Sea: Foundations of Algebraic Geometry. (Lecture notes).
[8]. Mumford, D. (1999). The Red Book of Varieties and Schemes. Springer

DOI: 10.48175/568