IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429
Volume 5, Issue 9, June 2025
Impact Factor: 7.67

On Generalized $U-$ Birecurrent Finsler Space in Berwald Sense

Abdalstar A. Saleem[1], Alaa A. Abdallah[*2] & Ammar Z. Hussein[3]
[1]Department. of Mathematics, Faculty of Sciences- Aden, University of Aden, Yemen
[2] Department of Mathematics, Education Faculty, University of Abyan, Zingibar, Yemen
[3] Department of Mathematics, Education Faculty, University of Lahej, Yafea, Yemen
saleemabdalstar@gmail.com, ala733.ala00@gmail.com & amaralalway@gmail.com

Abstract: *In this paper, we introduced a type of generalized birecurrent space in Berwald sense. The necessary and sufficient conditions for some tensors to be generalized birecurrent in Berwald sense have been obtained. Also, some results in the projection on indicatrix with respect to Cartan connection have been discussed.*

Keywords: Generalized birecurrent space, Berwald covariant derivative, Curvature tensor U^i_{jkh}, Douglas tensor D^i_{jkh}, Projection on indicatrix

I. INTRODUCTION

Finsler geometry, which extends the concepts of Riemannian geometry, provides a robust framework for analyzing spaces with direction-dependent (anisotropic) metric characteristics. In recent years, there has been increasing attention on recurrent Finsler structures, which are defined by the parallel transport of specific curvature tensors along geodesics. A key element in this study is Berwald's covariant differentiation, an essential technique within Finsler geometry. The Fundamentals and recent studies of Finsler geometry and the relationships between the curvature tensors in Finsler spaces discussed by [1, 4].

The generalized birecurrent Finsler spaces for some curvature tensors in Berwald sense have been studied by [3, 9, 13]. Also, the conditions for some tensors in Berwald sense have been obtained by [2, 5]. The curvature tensor U^i_{jkh} and relations it with several tensors in different spaces studied by [16-18].

Let us consider an $n-$ dimensional Finsler space $F_n$equipped with the line elements (x, y) and the fundamental metric function F positive homogeneous of degree one in y^i. The vectors y_i and y^i satisfy [12, 14]

(1.1) a) $y_i\, y^i = F^2$ and b) $\dot{\partial}_i y_j = \dot{\partial}_j y_i = g_{ij}$.

The fundamental metric tensor is satisfied [4, 10, 11, 15]

(1.2) a) $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2$, b) $g_{ij}(x,y)y^i = y_j$ and c) $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$.

The tensor C_{ijk} is homogeneous of degree -1 in y^i and symmetric in all its indices Berwald' s covariant derivative of the metric function F, metric tensor g_{ij}, vectors y^i and y_i, the unit vectors $l^i$and l_i are vanishing identically, i.e. [6, 8]

(1.3) a) $\mathcal{B}_k F = 0$, b) $\mathcal{B}_k g_{ij} = -2C_{ijk|h}y^h = -2y^h\mathcal{B}_k C_{ijk}$, c) $\mathcal{B}_k y^i = 0$,

d) $\mathcal{B}_k y_i = 0$, e) $\mathcal{B}_k l^i = 0$, f) $\mathcal{B}_k\, l_i = 0$, g) $l^i = \frac{y^i}{F}$,

h) $l_i = \frac{y_i}{F}$, i) $l_i = g_{ij} l^j$, j) $l_i l^i = 0$.

Berwald' s covariant derivative of an arbitrary tensor field T^i_h with respect to x^l is given by

(1.4) a) $\mathcal{B}_k T^i_h = \dot{\partial}_k T^i_h + T^r_h G^i_{rk} - T^i_r G^r_{sk} - (\dot{\partial}_r T^i_h) G^r_{hk}$.

And the commutation formula for the operator $\dot{\partial}_j$ and $\mathcal{B}_k$ are given by [14]

b) $\dot{\partial}_j \mathcal{B}_k T^i_h - \mathcal{B}_k \dot{\partial}_j T^i_h = T^r_h G^i_{jkr} - T^i_r G^r_{jkh}$.

K.Yano [19], defined the normal projective connection coefficients Π^i_{jk} by

IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429
Volume 5, Issue 9, June 2025
Impact Factor: 7.67

(1.5) a) $\Pi^i_{jk} = G^i_{jk} - \frac{1}{n+1} y^i G^r_{jkr}$ and b) $G^i_{jk} = \dot{\partial}_j G^i_k$.

The connection coefficients Π^i_{jk} is positively homogeneous of degree zero in y^i and symmetric in their lower indices. K. Yano [19] denoted this tensor by U^i_{jkh}. Thus

and b) $G^r_{jkhr} = \dot{\partial}_j G^r_{khr}$. $U^i_{jkh} = G^i_{jkh} - \frac{1}{n+1}(\delta^i_j G^r_{jkr} + y^i G^r_{jkhr})$) a(1.6)

The tensor U^i_{jkh} is called curvature tensor and G^i_{jkh} is connection of curvature tensor satisfies [17]

$G^i_{jkh} y^j = G^i_{kjh} y^j = G^i_{khj} y^j = 0.$ c) (1.6)

This tensor U^i_{jkh} is homogeneous of degree -1 in y^i and symmetric in its last two indices, i.e.

$U^i_{jkh} = U^i_{jhk}$.

Also this tensor satisfies the following [18]

(1.7) $U^r_{jrk} = U^r_{jkr} = G^r_{jkr}$,

1.8) $U^i_{jkh} y^j = 0,$ (

(1.9) $U^i_{jkh} y^h = U^i_{jhk} y^h = U^i_{jk}$, where $U^i_{jk} = \Pi^i_{jk}$.

The tensor U^i_{jk} is called torsion tensor and satisfies

(1.10) $U^i_{jk} = U^i_{kj}$,

(1.11) $U^r_{jr} = G^r_{jr}$,

(1.12) $U^i_{jk} y^k = U^i_{kj} y^k = G^i_j$,

where

(1.13) $G^i_j y^j = 2G^i$.

The tensor $U_{jk}$is called Ricci tensor satisfies the following [18]

(1.14) $U^r_{rkh} = U_{kh}$,

$U_{jk} = \frac{2}{n+1} G_{jk}$, (1.15)

where the tensor G_{jk} is components of the projective connection coefficient.

The Douglas tensor is given by [18]

(1.16) $D^i_{jkh} = U^i_{jkh} - \frac{1}{2}(\delta^i_j U_{kh} + \delta^i_k U_{jh})$.

Also this tensor satisfies the following:

(1.17) $D^i_{jkh} y^j = D^i_{Kjh} y^j = D^i_{khj} y^j = 0$.

Definition 1.1. The projection of any tensor T^i_j on indicatrix is given by [7]

(1.18) $p.T^i_j = T^\alpha_\beta h^i_\alpha h^\beta_j$,

where the angular metric tensor is homogeneous function of degree zero in y^i and defined by

(1.19) b) $h^i_j = \delta^i_j - l^i l_j$.

Definition 1.2. If the projection of tensor T^i_j on indicatrix I_{n-1} is the same tensor T^i_j, the tensor is called an indicatrix tensor or an indicatory tensor.

The projection of the vector y^i, the unit vector l^i and the metric tensor g_{ij} on indicatrix are given by [7]

(1.20) a) $p.y^i = 0$, b) $p.l^i = 0$, c) $p.g_{ij} = h_{ij}$, where d) $h_{ij} = g_{ij} - l_i l_j$.

Saleem [16] introduced the generalized $Ɓ_l U$ −recurrent Finsler space. *i.e.* the tensor U^i_{jkh} is characterized by the following condition:

(1.21) $Ɓ_l U^i_{jkh} = \lambda_l U^i_{jkh} + \mu_l(\delta^i_j g_{kh} + \delta^i_k g_{jh})$, $U^i_{jkh} \neq 0$.

Where λ_l and μ_l are non-zero covariant vector fields.

IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429 Volume 5, Issue 9, June 2025 Impact Factor: 7.67

II. An Generalized $\mathcal{B}_m\mathcal{B}_l U$-Birecurrent Space

Differentiating (1.21) convaraintly with respect to x^m in the sense of Berwald, we get

$$\mathcal{B}_m\mathcal{B}_l U^i_{jkh} = (\mathcal{B}_m\lambda_l + \lambda_m\lambda_l)U^i_{jkh} + (\mathcal{B}_m\mu_l + \lambda_l\mu_m + \mu_l\mathcal{B}_m)(\delta^i_j g_{kh} + \delta^i_k g_{jh}),$$

Above equation can be written as

(2.1) $$\mathcal{B}_m\mathcal{B}_l U^i_{jkh} = a_{lm}U^i_{jkh} + b_{lm}(\delta^i_j g_{kh} + \delta^i_k g_{jh}), \quad U^i_{jkh} \neq 0,$$

where $a_{lm} = \mathcal{B}_m\lambda_l + \lambda_m\lambda_l$ and $b_{lm} = \mathcal{B}_m\mu_l + \lambda_l\mu_m + \mu_l\mathcal{B}_m$ are non-zero of tensor fields of second order and called as birecurrence tensor fields of second order.

Definition 2.1. A Finsler space F_n which the normal projective curvature tensor U^i_{jkh} satisfies the condition (2.1) will be called *a generalized $\mathcal{B}U$ − birecurrent space* and denoted it briefly as $G(\mathcal{B}U) - BRF_n$.

Transvecting (2.1) by y^h, using (1.2b), (1.3c), (1.9), we get

(2.2) $$\mathcal{B}_m\mathcal{B}_l U^i_{jk} = a_{lm}U^i_{jk} + b_{lm}(\delta^i_j y_k + \delta^i_k y_j).$$

Contracting the indices i and j in (2.1) and using (1.14), we get

(2.3) $$\mathcal{B}_m\mathcal{B}_l U_{kh} = a_{lm}U_{kh} + b_{lm}(n+1)g_{kh}.$$

contracting the indices i and h in (2.1), we get in view of (1.7), the

(2.4) $$\mathcal{B}_m\mathcal{B}_l G^r_{jkr} = a_{lm}G^r_{jkr} + 2b_{lm}g_{jk}.$$

In view of (2.3) and (1.15), we get $G_{kh} = a_{lm}G_{kh} + \frac{1}{2}b_{lm}(n+1)^2 g_{kh}. \mathcal{B}_m\mathcal{B}_l(2.5)$

Transvecting (2.2) by y^k, using (1.12), (1.3c) and (1.1a), we get

(2.6) $$\mathcal{B}_m\mathcal{B}_l G^i_j = a_{lm}G^i_j + b_{lm}(\delta^i_j F^2 + y^i y_j).$$

Transvecting (2.6) by y^j and using (1.13), (1.1a) and (1.3c), we get

$\mathcal{B}_m\mathcal{B}_l G^i = a_{lm}G^i + 2b_{lm}y^i F^2.$ (2.7)

Contracting the indices i and k in (2.2) and using (1.11), we get

(2.8) $$\mathcal{B}_m\mathcal{B}_l G^r_{jr} = a_{lm}G^r_{jr} + b_{lm}(n+1)y_j.$$

Thus, we conclude

Theorem 2.1. *In $G(\mathcal{B}U) - BRF_n$, the torsion tensor U^i_{jk}, Ricci tensor U_{kh}, tensor G^r_{jkr}, Ricci tensor G_{kh}, deviation tensor G^i_j, vector G^i and the tensor G^r_{jr} are non-vanishing.*

Differentiating (1.16) covariantly twice with respect to $x^l$and x^m in the sense of Berwald, we get

(2.9) $$\mathcal{B}_m\mathcal{B}_l D^i_{jkh} = \mathcal{B}_m\mathcal{B}_l U^i_{jkh} - \frac{1}{2}(\delta^i_j\mathcal{B}_m\mathcal{B}_l U_{kh} + \delta^i_k\mathcal{B}_m\mathcal{B}_l U_{jh}).$$

Using (2.1) and (2.3) in (2.9), we get

(2.10) $$\mathcal{B}_m\mathcal{B}_l D^i_{jkh} = a_{lm}[(U^i_{jkh} - \frac{1}{2}(\delta^i_j U_{kh} + \delta^i_k U_{jh})] + \frac{1}{2}(1-n)b_{lm}(\delta^i_j g_{kh} + \delta^i_k g_{jh}).$$

Using (1.16) in (2.10), we get

(2.11) $$\mathcal{B}_m\mathcal{B}_l D^i_{jkh} = a_{lm}D^i_{jkh} + \eta_{lm}(\delta^i_j g_{kh} + \delta^i_k g_{jh}),$$

where $\eta_{lm} = \frac{1}{2}(1-n)b_{lm}$. Thus, we conclude

Theorem 2.2. *In $G(\mathcal{B}U) - BRF_n$, the Douglas tensor D^i_{jkh} is generalized birecurrent.*

If the Douglas tensor D^i_{jkh} is generalized birecurrent and Ricci tensor U_{kh} is behaves as birecurrent in a Finsler space, then the space is necessarily to be $G(\mathcal{B}U) - BRF_n$. Thus Eq. (2.9) can be written as

(2.12) $$\mathcal{B}_m\mathcal{B}_l U^i_{jkh} = \mathcal{B}_m\mathcal{B}_l D^i_{jkh} + \frac{1}{2}(\delta^i_j\mathcal{B}_m\mathcal{B}_l U_{kh} + \delta^i_k\mathcal{B}_m\mathcal{B}_l U_{jh}).$$

Using (2.3) and (2.11) in (2.12), we get

(2.13) $$\mathcal{B}_m\mathcal{B}_l U^i_{jkh} = a_{lm}[(D^i_{jkh} + \frac{1}{2}(\delta^i_j U_{kh} + \delta^i_k U_{jh})] + b_{lm}(\delta^i_j g_{kh} + \delta^i_k g_{jh}).$$

Using (1.16) in (2.13), we get

(2.14) $$\mathcal{B}_m\mathcal{B}_l U^i_{jkh} = a_{lm}U^i_{jkh} + b_{lm}(\delta^i_j g_{kh} + \delta^i_k g_{jh}).$$

Thus, we conclude

Theorem 2.3. *In a Finsler space F_n, if Douglas tensor D^i_{jkh} is generalized birecurrent and the Ricci tensor U_{kh} behaves as birecurrent, then the space considered is necessarily $G(\mathcal{B}U) - BRF_n$.*

III. The Necessary and Sufficient Conditions for Some Tensors to be Generalized Birrecurrent

We find the necessary and sufficient conditions for some tensors to be generalized birecurrent in $G(\mathcal{B}U) - BRF_n$. Let us consider $G(\mathcal{B}U) - BRF_n$ characterized by (2.1). Differentiating (2.4) partially with respect to y^h, using (1.2c), we get

(3.1) $\dot{\partial}_h \mathcal{B}_m \mathcal{B}_l G^r_{jkr} = (\dot{\partial}_h a_{lm}) G^r_{jkr} + a_{lm}(\dot{\partial}_h G^r_{jkhr}) + 2(\dot{\partial}_h b_{lm}) g_{jk} + 4 b_{lm} C_{hjk}.$

Using commutation formula exhibited by (1.4b) for G^r_{jkr}, using (1.2c), (1.6b) in (3.1), we get

(3.2) $\mathcal{B}_m(\dot{\partial}_h \mathcal{B}_l G^r_{jkr}) - (\mathcal{B}_s G^r_{jkr}) G^s_{hlm} - (\mathcal{B}_m G^r_{skr}) G^s_{hlj} - (\mathcal{B}_m G^r_{jsr}) G^s_{hlk}$
$= (\dot{\partial}_h a_{lm}) G^r_{jkr} + a_{lm} G^r_{jkhr} + 2(\dot{\partial}_h b_{lm}) g_{jk} + 4 b_{lm} C_{hjk}.$

Again, applying the commutation formula (1.4b) for G^r_{jkr}, using (1.6b) in (3.2), we get

(3.3) $\mathcal{B}_m \mathcal{B}_l G^r_{jkhr} - (\mathcal{B}_m G^r_{skr}) G^s_{hlj} - G^r_{skr}(\mathcal{B}_m G^s_{hlj}) - (\mathcal{B}_m G^r_{jsr}) G^s_{hlk} - G^r_{jsr}(\mathcal{B}_m G^s_{hlk}) - (\mathcal{B}_m G^r_{jsr}) G^s_{hlk} - (\mathcal{B}_s G^r_{jkr}) G^s_{hlm} - (\mathcal{B}_l G^r_{skr}) G^s_{hmj}$
$-(\mathcal{B}_l G^r_{jsr}) G^s_{hmk} = (\dot{\partial}_h a_{lm}) G^r_{jkr} + a_{lm} G^r_{jkhr} + 2(\dot{\partial}_h b_{lm}) g_{jk} + 4 b_{lm} C_{hjk}$

This shows that

(3.4) $\mathcal{B}_m \mathcal{B}_l G^r_{jkhr} = a_{lm} G^r_{jkhr}.$

If and only if

(3.5) $(\mathcal{B}_m G^r_{skr}) G^s_{hlj} + G^r_{skr}(\mathcal{B}_m G^s_{hlj}) + (\mathcal{B}_m G^r_{jsr}) G^s_{hlk} + G^r_{jsr}(\mathcal{B}_m G^s_{hlk})$
$+(\mathcal{B}_m G^r_{jsr}) G^s_{hlk} + (\mathcal{B}_s G^r_{jkr}) G^s_{hlm} + (\mathcal{B}_l G^r_{skr}) G^s_{hmj} + (\mathcal{B}_l G^r_{jsr}) G^s_{hmk} + (\dot{\partial}_h a_{lm}) G^r_{jkr} + 2(\dot{\partial}_h b_{lm}) g_{jk} + 4 b_{lm} C_{hjk}$
$= 0.$

Thus, we conclude

Theorem 3.1. *In $G(\mathcal{B}U) - BRF_n$, the tensor G^r_{jkhr} behaves as birecurrent if and only if (3.5) holds.*

Transvecting (3.3) by y^l, using (1.3c) and (1.6c), we get

(3.6) $y^l \mathcal{B}_m \mathcal{B}_l G^r_{jkhr} - y^l(\mathcal{B}_l G^r_{skr}) G^s_{hmj} - y^l(\mathcal{B}_l G^r_{jsr}) G^s_{hmk} = y^l(\dot{\partial}_h a_{lm}) G^r_{jkr}$
$+ y^l a_{lm} G^r_{jkhr} + 2 y^l(\dot{\partial}_h b_{lm}) g_{jk} + 4 y^l b_{lm} C_{hjk}.$

This shows that

(3.7) $y^l \mathcal{B}_m \mathcal{B}_l G^r_{jkhr} = y^l a_{lm} G^r_{jkhr}.$

If and only if

(3.8) $y^l(\mathcal{B}_l G^r_{skr}) G^s_{hmj} + y^l(\mathcal{B}_m G^r_{jsr}) G^s_{hmk} + y^l(\dot{\partial}_h a_{lm}) G^r_{jkr}$
$+2 y^l(\dot{\partial}_h b_{lm}) g_{jk} + 4 y^l b_{lm} C_{hjk} = 0.$

Thus, we conclude

Theorem 3.2. *In $G(\mathcal{B}U) - BRF_n$, the directional derivative of the tensor G^r_{jkhr} in directional of y^m is proportional to the tensor G^r_{jkhr} if and only if (3.8) holds.*

Again, transvecting (3.3) by y^m, using (1.3c) and (1.6c), we get

(3.9) $y^m \mathcal{B}_m \mathcal{B}_l G^r_{jkhr} - y^m G^r_{skr}(\mathcal{B}_m G^s_{hlj}) - y^m(\mathcal{B}_m G^r_{jsr}) G^s_{hlk} -$
$y^m G^r_{jsr}(\mathcal{B}_l G^s_{hlk}) - y^m(\mathcal{B}_s G^r_{jkr}) G^s_{hlk} = y^m(\dot{\partial}_h a_{lm}) G^r_{jkr} + a_{lm} y^m G^r_{jkhr}$
$+2 y^m(\dot{\partial}_h b_{lm}) g_{jk} + 4 y^m b_{lm} C_{hjk}.$

This shows that

(3.10) $y^m \mathcal{B}_m \mathcal{B}_l G^r_{jkhr} = a_{lm} y^m G^r_{jkhr} + y^m b_{lm} C_{hjk}.$

If and only if

(3.11) $y^m G^r_{skr}(\mathcal{B}_m G^s_{hlj}) + y^m(\mathcal{B}_m G^r_{jsr}) G^s_{hlk} + y^m G^r_{jsr}(\mathcal{B}_l G^s_{hlk})$
$+ y^m(\mathcal{B}_s G^r_{jkr}) G^s_{hlk} + y^m(\dot{\partial}_h a_{lm}) G^r_{jkr} + 2 y^m(\dot{\partial}_h b_{lm}) g_{jk} + y^m b_{lm} C_{hjk} = 0..$

Thus, we conclude

IJARSCT

International Journal of Advanced Research in Science, Communication and Technology

International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal

ISSN: 2581-9429 Volume 5, Issue 9, June 2025 Impact Factor: 7.67

Theorem 3.3. *In $G(\mathcal{B}U)-BRF_n$, the directional derivative of the tensor G^r_{jkhr} in the directional of y^l is proportional to the tensor G^r_{jkhr} if and only if (3.11) holds.*

Differentiating (1.6a) twice covariantly with respect to x^l and x^m in the sense of Berwald, we get

(3.12) $$\mathcal{B}_m\mathcal{B}_l U^i_{jkh} = \mathcal{B}_m\mathcal{B}_l G^i_{jkh} - \frac{1}{n+1}\left(\delta^i_j \mathcal{B}_m\mathcal{B}_l G^r_{jkr} + y^i \mathcal{B}_m\mathcal{B}_l G^r_{jkhr}\right).$$

Using (2.1) in (3.12), we get

(3.13) $$\mathrm{a}_{lm} U^i_{jkh} + \mathrm{b}_{lm}\left(\delta^i_j g_{kh} + \delta^i_k g_{jh}\right) = \mathcal{B}_m\mathcal{B}_l G^i_{jkh} - \frac{1}{n+1}\left(\delta^i_j \mathcal{B}_m\mathcal{B}_l G^r_{jkr} + y^i \mathcal{B}_m\mathcal{B}_l G^r_{jkhr}\right).$$

Using (1.6a), (3.3) and (2.4) in above equation, we get

(3.14) $$\mathcal{B}_m\mathcal{B}_l G^i_{jkh} - \mathrm{a}_{lm} G^i_{jkh} - \mathrm{b}_{lm}\left(\delta^i_j g_{kh} + \delta^i_k g_{jh}\right) = \frac{y^i}{n+1}\left[(\mathcal{B}_m G^r_{skr}) G^s_{hmj} + G^r_{skr}(\mathcal{B}_m G^s_{hlj}) + (\mathcal{B}_m G^r_{jsr}) G^s_{hlk} + G^r_{jsr}(\mathcal{B}_l G^s_{hlk}) + (\mathcal{B}_s G^r_{jkr}) G^s_{hlk} + (\mathcal{B}_s G^r_{jkr}) G^s_{hlm} + (\mathcal{B}_m G^r_{skr}) G^s_{hmj} + (\mathcal{B}_m G^r_{jsr}) G^s_{hmk} + (\dot{\partial}_h \mathrm{a}_{lm}) G^r_{jkr} + 2 g_{jk}(\dot{\partial}_h \mathrm{b}_{lm}) + 4\mathrm{b}_{lm} C_{hjk} + 2\mathrm{b}_{lm}\delta^i_j g_{jk}.\right.$$

This shows that

(3.15) $$\mathcal{B}_m\mathcal{B}_l G^i_{jkh} = \mathrm{a}_{lm} G^i_{jkh} + \mathrm{b}_{lm}\left(\delta^i_j g_{kh} + \delta^i_k g_{jh}\right)$$

if and only if

(3.16) $$(\mathcal{B}_m G^r_{skr}) G^s_{hmj} + G^r_{skr}(\mathcal{B}_m G^s_{hlj}) + (\mathcal{B}_m G^r_{jsr}) G^s_{hlk} + G^r_{jsr}(\mathcal{B}_l G^s_{hlk}) + (\mathcal{B}_s G^r_{jkr}) G^s_{hlk} + (\mathcal{B}_s G^r_{jkr}) G^s_{hlm} + (\mathcal{B}_m G^r_{skr}) G^s_{hmj} + (\mathcal{B}_m G^r_{jsr}) G^s_{hmk} + (\dot{\partial}_h \mathrm{a}_{lm}) G^r_{jkr} + (\dot{\partial}_h \mathrm{b}_{lm})(2 g_{jk}) + 2\mathrm{b}_{lm}\delta^i_j g_{jk} + 4\mathrm{b}_{lm} C_{hjk} = 0.$$

Thus, we conclude

Theorem 3.4. *In $G(\mathcal{B}U)-BRF_n$, the tensor G^i_{jkh} is generalized birecurrent if and only if (3.16) holds.*

Differentiating (1.5a) covariantly twice with respect to x^l and x^m in the sense of Berwald and using (1.3c), we get

(3.17) $$\mathcal{B}_m\mathcal{B}_l U^i_{jk} = \mathcal{B}_m\mathcal{B}_l G^i_{jk} - \frac{1}{n+1} y^i \mathcal{B}_m\mathcal{B}_l G^r_{jkr}.$$

Using (2.3), (2.4) and (1.5a) in (3.17), we get

(3.18) $$\mathcal{B}_m\mathcal{B}_l G^i_{jk} = a_{lm} G^i_{jk} + b_{lm}\left(\delta^i_j y_k + \delta^i_k y_j\right).$$

If and only if

(3.19) $$\frac{2y^i}{n+1} g_{jk} = 0.$$

Thus, we conclude

Theorem 3.5. *In $G(\mathcal{B}U)-BRF_n$, the tensor G^i_{jk} is non-vanishing if and only if (3.19) holds.*

IV. Projection on Indicatrix with respect to Berwald′s Connection

Let us consider a Finsler space F_n which the curvature tensor U^i_{jkh} is generalized birecurrent in the sense of Berwald, i.e. characterized by the condition (2.1). Now, in view of (1.18), the projection of the curvature tensor U^i_{jkh} on indicatrix is given by

(4.1) $$p.\,U^i_{jkh} = U^a_{bcd} h^i_a h^b_j h^c_k h^d_h.$$

Taking the covariant derivative of (4.1) with respect to x^l and x^m in the sense of Berwald and using the fact that $\mathcal{B}_l h^i_j = 0$, we get

(4.2) $$\mathcal{B}_m\mathcal{B}_l\left(p.\,U^i_{jkh}\right) = \mathcal{B}_m\mathcal{B}_l U^a_{bcd} h^i_a h^b_j h^c_k h^d_h.$$

Using (2.1) in (4.2), we get

(4.3) $$\mathcal{B}_m\mathcal{B}_l\left(p.\,U^i_{jkh}\right) = \left[a_{lm} U^a_{bcd} + b_{lm}\left(\delta^a_b g_{cd} + \delta^a_c g_{bd}\right)\right] h^i_a h^b_j h^c_k h^d_h.$$

In view of (1.20) and by using the fact that $\mathcal{B}_l h^i_j = 0$, Eq. (4.3) can be written as

IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429 Volume 5, Issue 9, June 2025 Impact Factor: 7.67

(4.4) $\mathcal{B}_m\mathcal{B}_l\left(p.U^i_{jkh}\right) = a_{lm}\left(p.U^i_{jkh}\right) + b_{lm}p.\left(\delta^i_j g_{kh} + \delta^i_k g_{jh}\right)$.

Thus, we conclude

Theorem 4.1. *The projection of the curvature tensor U^i_{jkh} of $G(\mathcal{B}U) - BRF_n$, on indicatrix is generalized birecurrent in the sense of Berwald.*

Let us consider a Finsler space for which the Douglas tensor $D^i_{jkh}$is generalized birecurrent in the sense of Berwald, i.e. characterized by (2.13). In view of (1.18), the projection of the Douglas tensor D^i_{jkh} on indicatrix is given by

(4.5) $p.D^i_{jkh} = D^a_{bcd}h^i_a h^b_j h^c_k h^d_h$.

Taking the covariant derivative of (4.5) with respect to x^l and x^m in the sense of Berwald and using the fact $\mathcal{B}_l h^i_j = 0$, we get

(4.6) $\mathcal{B}_m\mathcal{B}_l\left(p.D^i_{jkh}\right) = \mathcal{B}_m\mathcal{B}_l D^a_{bcd}h^i_a h^b_j h^c_k h^d_h$.

Using (2.11) in (4.6), we get

(4.7) $\mathcal{B}_m\mathcal{B}_l\left(p.D^i_{jkh}\right) = \left[a_{lm}D^i_{jkh} + \eta_{lm}(\delta^a_b g_{cd} + \delta^a_c g_{bd})\right]h^i_a h^b_j h^c_k h^d_h$.

In view of (1.18) and by using the fact $\mathcal{B}_l h^i_j = 0$, Eq. (4.7) can be written as

(4.8) $\mathcal{B}_m\mathcal{B}_l\left(p.D^i_{jkh}\right) = a_{lm}\left(p.D^i_{jkh}\right) + \eta_{lm}[p.(\delta^i_j g_{kh} + \delta^i_k g_{jh})]$.

Thus, we conclude

Theorem 4.2. *The projection of the Douglas tensor D^i_{jkh} of $G(\mathcal{B}U) - BRF_n$, on indicatrix is generalized birecurrent in the sense of Berwald.*

Let us consider a Finsler space which the projection of the curvature tensor U^i_{jkh} is generalized birecurrent in the sense of Berwald, i.e. characterized by (2.1). Using (1.18) in (2.1), we get

(4.9) $\mathcal{B}_m\mathcal{B}_l\left(U^a_{bcd}h^i_a h^b_j h^c_k h^d_h\right) = [a_{lm}(U^a_{bcd} + b_{lm}(\delta^a_b g_{cd} + \delta^a_c g_{bd})]h^i_a h^b_j h^c_k h^d_h$.

Using (1.19) in (4.9), we get

(4.10)
$$\mathcal{B}_m\mathcal{B}_l\{U^a_{bcd}(\delta^i_a - l^i l_a)(\delta^b_j - l^b l_j)(\delta^c_k - l^c l_k)(\delta^d_h - l^d l_h)\}$$
$$= a_{lm}\{U^a_{bcd}(\delta^i_a - l^i l_a)(\delta^b_j - l^b l_j)(\delta^c_k - l^c l_k)(\delta^d_h - l^d l_h)\}$$
$$+b_{lm}\{\delta^a_b g_{cd} + \delta^a_c g_{bd})(\delta^i_a - l^i l_a)(\delta^b_j - l^b l_j)(\delta^c_k - l^c l_k)(\delta^d_h - l^d l_h)\}.$$

Which can be written as

(4.11)
$$\mathcal{B}_m\mathcal{B}_l\big(U^i_{jkh} - U^i_{jkd}l^d l_h - U^i_{jch}l^c l_k + U^i_{jcd}l^c l_k l^d l_h - U^a_{jkh}l^i l_a\ ($$
$$+ U^a_{jkd}l^i l_a l^d l_h + U^a_{jch}l^i l_a l^c l_k - U^a_{jcd}l^i l_a l^c l_k l^d l_h\big)$$
$$= a_{lm}\big(U^i_{jkh} - U^i_{jkd}l^d l_h - U^i_{jch}l^c l_k + U^i_{jcd}l^c l_k l^d l_h - U^a_{jkh}l^i l_a$$
$$-U^a_{jkd}l^i l_a l^d l_h + U^a_{jch}l^i l_a l^c l_k - U^a_{jcd}l^i l_a l^c l_k l^d l_h\big)$$
$$+ b_{lm}[(\delta^i_j g_{kh} - \delta^i_j g_{kd}l^d l_h - \delta^i_j g_{ch}l^c l_k + \delta^i_j g_{cd}l^c l_k l^d l_h - \delta^i_b g_{kh}l^b l_j$$
$$+\delta^i_b g_{kd}l^b l_j l^d l_h + \delta^i_b g_{ch}l^b l_j l^c l_k - \delta^a_j g_{kh}l^i l_a - \delta^i_b g_{cd}l^b l_j l^c l_k l^d l_h$$
$$+\delta^a_j g_{kd}l^i l_a l^d l_h + \delta^a_j g_{ch}l^i l_a l^c l_k - \delta^a_j g_{cd}l^i l_a l^c l_k l^d l_h + \delta^a_b g_{kh}l^i l_a l^b l_j$$
$$-\delta^a_b g_{kd}l^i l_a l^b l_j l^d l_h - \delta^a_b g_{ch}l^i l_a l^b l_j l^d l_h + \delta^a_b g_{cd}l^i l_a l^b l_j l^c l_k l^d l_h)$$
$$+(\delta^i_k g_{jh} - \delta^i_k g_{jd}l^d l_h - \delta^i_c g_{jh}l^c l_k + \delta^i_c g_{jd}l^c l_k l^d l_h - \delta^i_k g_{bh}l^b l_j$$
$$+\delta^i_k g_{bd}l^b l_j l^d l_h + \delta^i_c g_{bh}l^b l_j l^c l_k - \delta^a_k g_{jh}l^i l_a - \delta^i_c g_{bd}l^b l_j l^c l_k l^d l_h$$
$$\delta^a_k g_{jd}l^i l_a l^d l_h + \delta^a_c g_{jh}l^i l_a l^c l_k - \delta^a_c g_{jd}l^i l_a l^c l_k l^d l_h + \delta^a_k g_{bh}l^i l_a l^b l_j$$
$$-\delta^a_k g_{bd}l^i l_a l^b l_j l^d l_h - \delta^a_c g_{bh}l^i l_a l^b l_j l^c l_k + \delta^a_c g_{bd}l^i l_a l^b l_j l^c l_k l^d l_h)].$$

Using (1.9), (3.17), (1.2b) and (1.3g) in (4.11), we get

(4.12)
$$\mathcal{B}_m\mathcal{B}_l\left(U^i_{jkh} - \tfrac{1}{F}U^i_{jk}l_h - \tfrac{1}{F}U^i_{jh}l_k - U^a_{jkh}l^i l_a + U^a_{jk}l^i l_a l_h + \tfrac{1}{F}U^a_{jh}l^i l_a l_k\right)$$
$$= a_{lm}\left(U^i_{jkh} - \frac{1}{F}U^i_{jk}l_h - \frac{1}{F}U^i_{jh}l_k - U^a_{jkh}l^i l_a + U^a_{jk}l^i l_a l_h + \frac{1}{F}U^a_{jh}l^i l_a l_k\right)$$
$$+b_{lm}\{[\delta^i_j g_{kh} - \tfrac{1}{F}\delta^i_j y_k l_h - \tfrac{1}{F}\delta^i_j y_h l_k - \delta^a_j g_{kh}l^i l_a + \delta^a_j y_k l^i l_a l_h + \delta^a_j y_h l^i l_a l_k]$$

IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429
Volume 5, Issue 9, June 2025
Impact Factor: 7.67

$$+[\delta^i_k g_{jh} - \frac{1}{F}\delta^i_k y_j l_h - \frac{1}{F}\delta^i_h y_j l_k - \delta^a_k g_{jh} l^i l_a + \delta^a_k y_j l^i l_a l_h + \delta^a_h y_j l^i l_a l_k]\}.$$

Since the torsion tensor U^i_{jk} is given by (2.2) and in view of (2.2), (1.3a) and (1.3c), the eq. (4.12) can be written as

$$(4.13)\qquad \mathcal{B}_m\mathcal{B}_l(U^i_{jkh} - U^a_{jkh} l^i l_a) = [a_{lm}U^i_{jkh} + b_{lm}(\delta^i_j g_{kh} - \delta^i_k g_{jh})]$$
$$-[a_{lm}U^a_{jkh} + b_{lm}(\delta^i_j g_{kh} - \delta^i_k g_{jh})]l^i l_a.$$

Thus, we conclude

Theorem 4.3. *If the projection of the tensor $(U^i_{jkh} - U^a_{jkh} l^i l_a)$ on indicatrix is generalized birecurrent, then the space is $G(\mathcal{B}U) - BRF_n$.*

Remark 4.1. *In $G(\mathcal{B}U) - BRF_n$, the projection of the U^i_{jkh} on indicatrix is generalized birecurrent, if and only if $U^a_{jkh} l^i l_a$ is generalized birecurrent.*

Let us consider a Finsler space F_n which the projection of the Douglas tensor D^i_{jkh} on indicatrix is generalized birecurrent i.e. characterized by (2.11). Using (1.18) in (2.11), we get

$$(4.14)\qquad \mathcal{B}_m\mathcal{B}_l D^a_{bcd} h^i_a h^b_j h^c_k h^d_h = [a_{lm}D^a_{bcd} + \eta_{lm}(\delta^a_b g_{cd} + \delta^a_c\, g_{bd})]h^i_a h^b_j h^c_k h^d_h$$

Using (1.19) in (5.14), we get

$$(4.15)\qquad \mathcal{B}_m\mathcal{B}_l\{D^a_{bcd}(\delta^i_a - l^i l_a)(\delta^b_j - l^b l_j)(\delta^c_k - l^c l_k)(\delta^d_h - l^d l_h)\}$$
$$= a_{lm}\{D^a_{bcd}(\delta^i_a - l^i l_a)(\delta^b_j - l^b l_j)(\delta^c_k - l^c l_k)(\delta^d_h - l^d l_h)\}$$
$$+\eta_{lm}\{(\delta^i_j g_{cd} + \delta^i_k\, g_{bd})(\delta^i_a - l^i l_a)(\delta^b_j - l^b l_j)(\delta^d_h - l^d l_h)(\delta^d_h - l^d l_h)\}.$$

In view of (4.15), using (1.3e), we get

$$(4.16)\qquad \mathcal{B}_m\mathcal{B}_l(D^i_{jkh} - D^a_{jkh} l^i l_a) = [a_{lm}D^i_{jkh} + \eta_{lm}(\delta^i_j g_{kh} - \delta^i_k g_{jh})]$$
$$-[a_{lm}D^a_{jkh} + \eta_{lm}(\delta^i_j g_{kh} - \delta^i_k g_{jh})]l^i l_a.$$

Thus, we conclude

Theorem 4.4. *If the projection of the tensor $(D^i_{jkh} - D^a_{jkh} l^i l_a)$ on indicatrix is generalized birecurrent, then the space is $G(\mathcal{B}U) - BRF_n$.*

Corollary 4.2. *In $G(\mathcal{B}U) - BRF_n$ the projection of the D^i_{jkh} on indicatrix is birecurrent, if and only if $D^a_{jkh} l^i l_a$ is generalized birecurrent.*

V. Conclusion

We studied different tensors which satisfy the generalized birecurrent property in Berwald sense. The necessary and sufficient conditions for the Douglas tensor □□□□ to be generalized birecurrent have been obtained. Also, we discussed the projection on indicatrix in sense of Berwald for some tensors which behave as generalized birecurrent in $G(\mathcal{B}U) - BRF_n$.

REFERENCES

[1]. Abdallah A. A., Study on the relationship between two curvature tensors in Finsler spaces, Journal of Mathematical Analysis and Modeling, 4(2), 112-120, (2023).

[2]. Abdallah A. A., Navlekar A. A. and Ghadle K. P., The necessary and sufficient condition for some tensors which satisfy a generalized $\mathcal{B}P$ −recurrent Finsler space, International Journal of Scientific and Engineering Research, 10(11), 135-140, (2019).

[3]. Abdallah A. A., Navlekar A. A. and Ghadle K. P., On certain generalized $\mathcal{B}P$ −birecurrent Finsler space, Journal of International Academy of Physical Sciences, 25(1), 63-82, (2021).

[4]. Abdallah A. A., Navlekar A. A., Ghadle K. P. and Hardan B., Fundamentals and recent studies of Finsler geometry, International Journal of Advances in Applied Mathematics and Mechanics, 10(2), 27-38, (2022).

[5]. Ahmed F. A. and Abdallah A. A., The conditions for various tensors to be generalized $\mathcal{B}$ −trirecurrent tensor, International Journal of Advanced Research in Science, Communication and Technology, 4(2), 511-517, (2024).

[6]. Ghadle K. P., Navlekar A. A. and Abdallah A. A., On $\mathcal{B}$ −covariant derivative of first order for some tensors in different spaces, Journal of Mathematical Analysis and Modeling, 2(2), 30-37, (2021).

[7]. Gheorghe M., The indicatrix in Finsler geometry, Analele Stiintifice Ale Uuiversității Matematică. Tomul LIII, 163-180, (2007).

[8]. Izumi H., On P^*-Finsler space I, II, Memo. Defence Acad. (Japan), 16, 133-138, (1977).

[9]. Hamoud A. A., Navlekar A. A., Abdallah A. A. and Ghadle K. P., Decomposition for Cartan's second curvature tensor of different order in Finsler spaces, Nonlinear Functional Analysis and Applications, 27(2), 433-448, (2022).

[10]. Hamoud A. A., Navlekar A. A., Abdallah A. A. and Ghadle K. P., On special spaces of $h(hv)$ −torsion tensor C_{jkh} in generalized recurrent Finsler space, Bull. Pure Appl. Sci. Sect. E Math. Stat. 41E (1), 74-80, (2022).

[11]. Matsumoto M, Remarks on Berwald and Landsberg Spaces, Contemporary Math. 195, 79-81, (1996).

[12]. Qasem F. Y., On transformation in Finsler spaces, D.Phil. Thesis, University of Allahabad, Allahabad (India), (2000).

[13]. Qasem F. Y. and Hadi W. H., On a generalized βR- birecurrent Finsler spaces, American Sci. Res. J. of Engi. Tech. and Sci. 19(1), 9-18, (2016).

[14]. Rund H. (1959); 2nd (in Russian), Nauka, Moscow, The differential geometry of Finsler spaces, Springer-verlag, Berlin Göttingen-Heidelberg, (1981).

[15]. Saleem A. A. and Abdallah A. A., Certain identities of C^h in Finsler spaces, International Journal of Advanced Research in Science, Communication and Technology, 3(2), 620-622, (2023).

[16]. Saleem A. A., On a generalized $\mathcal{B}_m U$ - recurrent Finsler space, University of Aden Journal of Natural and Applied Sciences, 23(2), 457-462, (2019).

[17]. Saleem A. A. and Abdallah A. A., On U - recurrent Finsler space, Int. R. J. of. Inn. in. Eng. And Tech. 6(1), 58 – 63, (2022).

[18]. Saleem A. A. and Abdallah A. A., Study on U^h - birecurrent Finsler space, IJARSCT. 2(3), 28 – 39, (2022).

[19]. Yano K., The theory of lie-derivatives and its applications. North – Holland publishing Co. Amsterdam, (1957).