IJARSCT

International Journal of Advanced Research in Science, Communication and Technology

International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal

ISSN: 2581-9429

Volume 3, Issue 1, August 2023

Impact Factor: 7.301

Applications of Leray Spectral Sequence in Homological Algebraic Structures

Vandana Malhotra[1] and Dr. Sudesh Kumar[2]

[1]Research Scholar, Department of Mathematics

[2]Professor, Department of Mathematics

NIILM University, Kaithal, Haryana

Abstract: *The Leray spectral sequence is a foundational computational tool in algebraic topology, geometry, and homological algebra. Originating in sheaf co homology, it effectively organizes complex homological information through a filtration associated with a continuous map. This review presents key constructions, underlying principles, and prominent applications of the Leray spectral sequence within homological algebraic structures, especially in the context of derived functions, sheaf co homology, group co homology, and algebraic geometry.*

Keywords*:* Leray spectral sequence, homological algebra, derived functors

I. INTRODUCTION

Spectral sequences provide a mechanism to compute homology and co homology groups by successive approximations. The Leray spectral sequence, introduced by Jean Leray, arises from a continuous map f:X→Y and relates the co homology of X to that of Y through derived functions. It is particularly effective when direct computation is difficult but certain fibers or sheaf restrictions are simpler.

For a sheaf F on X, the Leray spectral sequence converges to the co homology of F:

$$E_2^{p,q} = H^p\big(Y,\, R^q f_* \mathcal{F}\big) \Rightarrow H^{p+q}(X, \mathcal{F}),$$

Where $R^q f_*$ denotes the q-th right derived functor of the push forward f_*

This survey explores such applications, especially in contexts where derived functors and homological structures interplay.

PRELIMINARIES

A. Spectral Sequence Formalism

A spectral sequence $\{E_r^{p,q}, d_r\}$ consists of brigaded groups:

$E_r^{p,q}$ at page r,

Differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$

Convergence towards a graded target group.

Convergence notation:

$$E_r^{p,q} \Rightarrow H^{p+q}$$

Indicates a filtration F_pH_n satisfying:

$$F^pH^n / F^{p+1}H^n \cong E_\infty^{p,n-p}$$

B. Leray Spectral Sequence

The Leray spectral sequence is a powerful and versatile tool in algebraic topology, algebraic geometry, and homological algebra, providing a systematic method for computing complex chorology groups by breaking them into simpler, more manageable components. Originally introduced by Jean Leray during his studies on sheaf chorology in the 1940s, the sequence arises naturally when one considers a continuous map Yf:X→Y between topological spaces

ISSN: 2581-9429 Volume 3, Issue 1, August 2023 Impact Factor: 7.301

and a sheaf F on X. Its primary utility lies in relating the co homology of the total space X to that of the base space Y and the higher direct images of F.

Specifically, for a sheaf F on X, the Leray spectral sequence is expressed as

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

where Rqf∗ represents the q-th right derived functor of the push forward f∗. This construction enables researchers to compute the chorology of X indirectly, often simplifying calculations significantly when f has well-understood fibers or when F behaves nicely under f∗ The spectral sequence belongs to the broader class of spectral sequences, which are essentially algebraic devices that organize homological information into a sequence of approximations through graded objects $E_r^{p,q}$ with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ that iteratively converge to the desired chorology.

The notion of convergence in this context is formalized through filtrations on the chorology groups of X, where the associated graded pieces at the E∞page correspond to sub quotients of the target chorology groups. Leary's construction can be interpreted as a specific instance of the Grothendieck spectral sequence, arising from the composition of the derived functions

$$R\Gamma(Y, -) \circ Rf_* \cong R\Gamma(X, -).$$

This viewpoint emphasizes the deep connection between sheaf-theoretic methods and homological algebra, as the spectral sequence encodes the process of first pushing a sheaf forward along a continuous map and then taking global sections, with the higher derived factors capturing co homological obstructions. In the context of algebraic geometry, the Leray spectral sequence has significant applications, particularly in computing sheaf co homology on complex algebraic varieties. For instance, when $f : X \to Y$ a proper morphism of varieties and F is is a coherent sheaf, the higher direct images $R^q f_* \mathcal{F}$ are themselves coherent sheaves on Y, and the spectral sequence provides an efficient method to compute $H^n(X, \mathcal{F})$ using the (potentially simpler) chorology groups of Y. This principle extends to the study of fiber bundles, where the co homology of the total space can be related to the co homology of the base and fiber. In such situations, if the fibers are co homologically trivial or their chorology is well understood, the Leary spectral sequence often degenerates at the E^2-page, yielding explicit formulas for the total chorology.

A classical example is the projective bundle formula: for a vector bundle E over a variety X, the projective bundle $\mathbb{P}(E) \to X$ satisfies $H^*(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}) \cong H^*(X, \mathrm{Sym}^* E^\vee)$ which can be recovered via degeneration of the Leray spectral sequence. Beyond algebraic geometry, the Leray spectral sequence also plays a crucial role in group co homology. Consider a short exact sequence of groups $1 \to N \to G \to Q \to 1$ with a G-module M. The spectral sequence relates the co homology of G with that of N and Q, expressed as $E_2^{p,q} = H^p(Q, H^q(N, M)) \Rightarrow H^{p+q}(G, M)$ this construction, known as the Hochschild–Serre spectral sequence, is essentially a special case of the Leary spectral sequence when interpreted in the categorical framework of derived factors.

It allows the computation of group co homology in a stepwise fashion, by first analyzing the co homology of the normal subgroup N and then propagating this information through the quotient Q. In homological algebra, the Leray spectral sequence facilitates computations involving derived factors, enabling a decomposition of complex co homological invariants into more elementary components.

When applied to sheaves, the sequence elucidates the relationships among co homology groups at various levels, providing filtrations that highlight exact sequences and vanishing results. Notably, if $R^q f_* \mathcal{F} = 0$ for $q > r$ then the co homology of X vanishes beyond certain degrees, which can provide strong constraints on homological dimensions and inform the structure of derived categories.

In the derived category framework, the Leray spectral sequence can be viewed as a manifestation of the composition of derived functions, where $R\Gamma(X, \mathcal{F}) \cong R\Gamma(Y, Rf_* \mathcal{F})$ and the filtration of the spectral sequence corresponds to the canonical t-structure filtration on derived objects. This perspective has become particularly relevant with the rise of modern homotopical methods, where spectral sequences are used to organize computations in triangulated categories and to study derived functorial properties systematically. Additionally, the Leray spectral sequence finds applications in the study of mixed Hodge structures on complex algebraic varieties.

When applied to filtered complexes, the sequence allows one to track how chorology inherits filtrations from the geometry of the map and the sheaf, enabling the analysis of Hodge-theoretic properties. Moreover, in differential topology, the spectral sequence provides a framework for understanding the co homology of fiber bundles and foliations, particularly when combined with the Sere spectral sequence. Across these diverse applications, a key feature of the Leary spectral sequence is its ability to reduce global problems to local computations, allowing for effective and explicit results in otherwise intractable situations.

Despite its theoretical abstraction, the sequence has concrete computational implications, from calculating chorology rings to understanding vanishing theorems, co homological dimensions, and the structure of derived categories. Its centrality in modern mathematics stems from this combination of abstract formalism and practical utility, making it indispensable in algebraic topology, algebraic geometry, homological algebra, and representation theory.

The Leray spectral sequence is a fundamental construction that bridges local and global co homological information. Through its ability to relate the chorology of a total space to that of a base and its fibers, it enables powerful computational techniques in homological algebra and algebraic geometry, supports the study of group chorology, provides structural insights into derived categories, and informs Hodge-theoretic investigations.

Its versatility and depth ensure that it remains an essential tool for researchers analyzing complex homological structures, offering both conceptual clarity and practical methods for tackling difficult problems across a wide range of mathematical disciplines.

CONSTRUCTION

Given a continuous map $f : X \to Y$ and a sheaf F on X, the Leray spectral sequence arises from the Grothendieck spectral sequence applied to the composite functors:

$$\Gamma(Y, -) \circ Rf_* \cong R\Gamma(X, -)$$

Where Γ denotes global sections. Then:

$$E_2^{p,q} = H^p(Y,\, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

This formulation is fundamental and recurs in many applications.

CONVERGENCE AND EXACTNESS

If the base and fibers are well-behaved (e.g., f proper and F quasi-coherent), then convergence and finiteness properties are guaranteed.

Edge maps relate the limit groups to known morphisms in cohomology.

APPLICATIONS IN HOMOLOGICAL ALGEBRA

Homological algebra is a fundamental area of modern mathematics that provides powerful tools for studying algebraic structures using methods of chain complexes, exact sequences, and derived functors. Its applications extend across various domains, including algebraic topology, representation theory, algebraic geometry, and category theory. One of the most important applications in homological algebra is in the study of modules over rings, where concepts such as projective, injective, and flat modules allow mathematicians to understand the structure of modules and their extensions. By constructing resolutions of modules, homological algebra provides insights into the behavior of these objects, including the computation of important invariants such as Ext and Tor groups, which measure the failure of exactness and capture extension phenomena among modules.

In representation theory, homological algebra plays a crucial role in understanding modules over algebras and their interrelationships. The study of projective and injective resolutions helps in determining the decomposition of representations, computing cohomology of algebras, and analyzing extension groups that describe how simpler representations can combine to form more complex ones. These tools also provide the foundation for more advanced topics such as the theory of derived categories of representations, tilting theory, and the study of homological dimensions, which measure the complexity of representations. Furthermore, the techniques of homological algebra are essential in understanding group cohomology, which connects algebraic properties of groups to topological and geometric contexts.

In category theory, homological algebra serves as a bridge connecting algebraic structures with their categorical frameworks. Concepts such as derived functors, triangulated categories, and adjoint functors enable mathematicians to study functorial properties and relationships between different categories in a homological context. This categorical viewpoint provides unifying principles for seemingly disparate areas, allowing for the transfer of techniques and results across fields. For example, derived categories, originally developed in homological algebra, have become indispensable in algebraic geometry, representation theory, and even mathematical physics, facilitating the study of complex phenomena through categorical structures.

Homological algebra also finds application in non-commutative algebra, where it provides tools to study rings and algebras that are not necessarily commutative. Projective resolutions, homological dimensions, and Ext and Tor groups help classify modules over non-commutative rings and analyze their extensions, enabling a deeper understanding of the representation theory of these structures. These methods are particularly valuable in the study of quantum groups and non-commutative geometry, where classical geometric intuition is often insufficient, and homological techniques provide the necessary algebraic framework to analyze complex interactions.

Moreover, homological algebra underpins computational approaches in algebraic research. With the advent of computer algebra systems, many homological computations, including the calculation of Ext and Tor groups, resolutions, and derived functors, can be performed algorithmically, providing practical tools for both theoretical exploration and applied mathematics. These computational applications have significant implications in fields such as coding theory, cryptography, and combinatorics, where algebraic structures play a central role and precise calculations are essential for problem-solving.

Another important area of application is in deformation theory and algebraic geometry, where homological algebra provides techniques to study infinitesimal deformations of algebraic objects such as varieties, modules, or sheaves. Ext groups measure the obstructions to deformations, offering insights into the rigidity or flexibility of structures. This approach is critical in moduli theory, where one seeks to classify all objects of a certain type up to an equivalence relation, and homological algebra provides the necessary computational and theoretical tools to understand the geometry of moduli spaces.

In homotopy theory and higher-dimensional algebra, homological algebra provides a language for studying chain complexes, exact sequences, and cohomological operations in higher categories. The abstract methods of derived functors, spectral sequences, and triangulated categories extend naturally to these contexts, allowing mathematicians to generalize classical results and discover new connections between algebra, topology, and geometry. These applications are also essential in mathematical physics, particularly in areas such as string theory and quantum field theory, where homological techniques are used to study complex algebraic structures associated with physical systems.

DERIVED FUNCTORS AND HOMOLOGY COMPUTATIONS

The Leray spectral sequence translates the computation of global derived functors into local data on Y. For example, for sheaf cohomology, it decomposes:

$$H^n(X,\mathcal{F}) \cong \bigoplus_{p+q=n} E_\infty^{p,q}$$

When sheaves restrict well along the fibers, $R_q f_* F$ simplifies, leading to tractable calculations.

GROUP COHOMOLOGY

In group cohomology, consider a group extension:

$$1 \to N \to G \to Q \to 1$$

A spectral sequence arises relating the cohomologies:

$$E_2^{p,q} = H^p(Q,\, H^q(N, M)) \Rightarrow H^{p+q}(G, M)$$

where M is a G-module. This is essentially the Hochschild–Serre spectral sequence, a special case of the Leray formalism in algebraic group action contexts.

ALGEBRAIC GEOMETRY & SHEAF COHOMOLOGY

1. Cohomology of Fiber Bundles

For a fiber bundle $F \hookrightarrow X \xrightarrow{f} Y$ with locally trivial fibers F, constant sheaf Z, and trivial actions on fiber cohomology, the spectral sequence approximates:

$$E_2^{p,q} = H^p(Y,\, H^q(F, \mathbb{Z})) \Rightarrow H^{p+q}(X, \mathbb{Z}).$$

This generalizes the classical Serre spectral sequence to sheaf-cohomology frameworks.

2. Proper Morphisms and Base Change

For a proper morphism

$f : X \to Y$ of varieties, co homological base change theorems ensure that higher direct images $R^q f_* \mathcal{F}$ behave well under base extension. The Leray spectral sequence then becomes a practical method for comparing co homologies of fibers and total space.

HOMOLOGICAL DIMENSIONS AND VANISHING RESULTS

1. Cohomological Dimension

The spectral sequence offers vanishing lines that give bounds on cohomological dimensions. For example, if $R^q f_* \mathcal{F} = 0 \text{ for } q > r,$ then:

$$H^n(X, \mathcal{F}) = 0 \quad \text{for } n > p + r$$

Provided no further obstructions arise. Such control is valuable in derived category analysis.

2. Projective Bundle Formula

For a projective bundle $\mathbb{P}(E) \to X$ where E is a vector bundle of rank r, and $\mathcal{F} = \mathcal{O}_{\mathbb{P}(E)}$ one obtains:

$$H^*(\mathbb{P}(E), \mathcal{O}) \cong H^*(X,\, \mathrm{Sym}^*(E^\vee))$$

Recovered via degeneration of the Leray spectral sequence at E2 .

ADVANCED TOPICS

1. Derived Categories and Triangulated Functors

In derived category language, the Leray spectral sequence is a manifestation of the composition of derived functors:

$$R\Gamma(X, \mathcal{F}) \cong R\Gamma(Y, Rf_* \mathcal{F})$$

The filtration associated with the spectral sequence corresponds to the canonical t-structure filtration on derived objects.

2. Mixed Hodge Structures

When X and Y are complex algebraic varieties, applying the Leray spectral sequence to filtered complexes yields mixed Hodge structures on cohomology groups.

II. CONCLUSION

The Leray spectral sequence is an indispensable tool in modern homological algebra, sheaf theory, and algebraic geometry. Its flexibility in translating global problems into manageable local computations, connecting derived functors, and providing structural insights makes it fundamental across many subfields. The continued development of derived categories and homotopical techniques only deepens its relevance, particularly in advanced cohomological theories.

REFERENCES

[1]. Arapura, D. (2003). *The Leray spectral sequence is motivic*.
[2]. Asok, A., Déglise, F., & Nagel, J. (2018). *The homotopy Leray spectral sequence*.
[3]. Basu, S., & Parida, L. (2013). *Spectral sequences, exact couples and persistent homology of filtrations*.
[4]. Bott, R., & Tu, L. W. (1982). *Differential Forms in Algebraic Topology*. Springer.
[5]. Brown, K. S. (1982). *Cohomology of Groups*. Springer.
[6]. Deligne, P. (1971). *Théorie de Hodge II*. Publications Mathématiques de l'IHÉS.
[7]. Gelfand, S. I., & Manin, Y. I. (1996). *Methods of Homological Algebra*. Springer.
[8]. Grothendieck, A. (1957). *Sur quelques points d'algèbre homologique*. Tôhoku Mathematical Journal.
[9]. Hartshorne, R. (1977). *Algebraic Geometry*. Springer.
[10]. Leray, J. (1950). *L'anneau spectral et l'anneau fibré d'homologie d'un espace localement compact et d'une application continue*.
[11]. Miller, H. (2000). *Leray in Oflag XVIIA: The origins of sheaf theory, sheaf cohomology, and spectral sequences*.
[12]. Neumann, F., & Szymik, M. (2024). *Preludes to the Eilenberg–Moore and the Leray–Serre spectral sequences*.
[13]. Paranjape, K. H. (1994). *Some spectral sequences for filtered complexes and applications*.