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Generalized Cesaro Sequence Spaces and Their K¨othe–Toeplitz Duals

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Abstract: *In this paper we have constructed a generalized Cesaro sequence spaces of absolute type and determined its Köthe-Toeplitz duals in the operator form and found some interesting results.*

Keywords: *Köthe-Toeplitz*

I. INTRODUCTION

Cesaro sequence spaces of absolute type and non-absolute type were defined and studied respectively in [9] and [7]. Ahmad, Z.U. and Mursaleen defined Cesaro sequence spaces of a bounded type and determined its Ko¨the-Toeplitz duals in [1].

Lorentz has introduced the concept of an almost convergent sequence in [3]. An equivalent condition for almost convergence proved by him is that a bounded sequence (xk) of complex numbers is almost convergent iff the sequence

$$t_{ni} = \frac{x_{n+1} + x_{n+2} + \cdots + x_{n+i}}{i}, \quad n = 1, 2, 3, \ldots$$

converges uniformly in n as i → ∞. Taking this condition we have defined Lorentz sequence spaces of a bounded type and determined its Ko ¨ the-Toeplitz duals. The Lorentz sequence spaces so constructed is more general then the one constructed in [1].

Besides, we have constructed in this paper the generalized cesaro sequence spaces of absolute type and determined its Ko¨the-Toeplitz duals in the op- erator form.

For K¨othe-Toeplitz duals of complex valued or scalar valued sequence spaces, we refer to [2] and for a generalized version in the operator form we refer to [4].

II. LORENTZ SEQUENCE SPACES OF A BOUNDED TYPE

Let ω denote the vector space of all complex valued sequences x = (x_k). For a fixed positive number n and 0 < p < 1, we define

$$z(p,n) = \left\{ x = (x_k) \in \omega : \sup_{k\geq 1} \left| \frac{1}{k} \sum_{i=1}^{k} x_{n+i} \right|^p < \infty \right\} 2.1$$

and

$$z(p) = \left\{ x = (x_k) \in \omega : \sup_{k\geq 1} \left| \frac{1}{k} \sum_{i=1}^{k} x_{n+i} \right|^p < \infty, n \geq 0 \right\} 2.2$$

If we put n = 0 in (2.1), then $z(p, n)$ is the same as the Cesaro sequence spaces of a bounded type constructed in [1]. The spaces $z(p, n)$ and $z(p)$ have been studied in [6] by Nanda, *S* and Mohanty, *S*.

If we define for $x \in z(p, n)$

$$\rho(x)=\sup_{k\geq 1}\left|\frac{1}{k}\sum_{i=1}^{k}x_{n+i}\right|^{p},$$

the ρ is a p-semi norm on z(p, n). For p-seminorm we refer to [5]. However, for $x \in z(p)$, if we define

$$\|x\|=\sup_{k\geq 1 n\geq 0}\left|\frac{1}{k}\sum_{i=1}^{k}x_{n+i}\right|^{p},$$

then || x || is a well defined p-norm on z(*p*).

In the following, we prove that z(p) is a complete p-normed space and we have determined that its β-dual $z\beta(p)$ is $D(p) \cap D0$, where

$$z^{\beta}(p)=\left\{a=(a_k)\in\omega:\sum_{k=1}^{\infty}a_kx_k\ converges\ for\ every\ x=(x_k)\in z(p)\right\}2.3.$$

$$D(p)=\left\{a=(a_k)\in\omega:\sum_{k=1}^{\infty}k\left|\Delta a_k\right|N^{\frac{1}{p}}\ converges\ for\ all\ N\geq 1\right\}2.4$$

$$D_0=\left\{a=(a_k)\in\omega:\lim_{k\to\infty}ka_k=0\right\}\ \ 2.5$$

where $\Delta a_k = a_k - a_{k+1}$

Theorem 2.1: Z(p) is a complete p-normed space.

Proof: Let $\{x^r\}_{r=1}^{\infty}$ be a Cauchy sequence in Z(p) where

$$x^r=(x_i^r)_{i=1}^{\infty},\quad r=1,2,3,\ldots$$

Given $0 < \varepsilon < 1$, there exists a positive integer N such that

$$\|x^r-x^s\|<\varepsilon,\quad\forall\, r,s\geq N$$

$$\Rightarrow\sup_{k\geq 1}\left|\frac{1}{k}\sum_{i=1}^{k}\left(x_{n+i}^r-x_{n+i}^s\right)\right|^{p}<\varepsilon,\quad\forall r,s\geq N$$

$$\Rightarrow|x_{n+i}^r-x_{n+i}^s|^p<\varepsilon^{1/p}$$

$$<\varepsilon,\quad\forall n\geq 0,\ r,s\geq N\ 2.6$$

This shows that for a fixed i $(1\leq i<\infty)$ the sequence $\{x_i^r\}_{r=1}^{\infty}$ is a Cauchy sequence of complex numbers.

Since the space of complex numbers is complete, therefore $\{x_i^r\}_{r=1}^{\infty}$ converges in it. Let

$$x_i^r\to x_i\quad as\ r\to\infty$$

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Define $x = (x_i)_{i=1}^{\infty}$ and taking limit $s \to \infty$ in (2.6), we get

$$|x^r_{n+i} - x_{n+i}|^p < \varepsilon^{1/p}$$

$$< \varepsilon, \quad \forall n \geq 0,\ r \geq N$$

Therefore, we have

$$\sup_{k\geq 1}\left|\frac{1}{k}\sum_{i=1}^{k}(x^r_{n+i}-x_{n+i})\right|^p \leq \varepsilon, \quad \forall r \geq N$$

or

$$\|x^r - x\| \leq \varepsilon, \quad \forall r \geq N$$

or

$$x^r \to x \quad asr \to \infty$$

Now

$$\|x\| = \sup_{k\geq 1}\left|\frac{1}{k}\sum_{i=1}^{k}x_{n+i}\right|^p$$

$$\leq \sup_{k\geq 1 n\geq 0}\left|\frac{1}{k}\sum_{i=1}^{k}(x_{n+i}-x^r_{n+i})\right|^p + \sup_{k\geq 1 n\geq 0}\left|\frac{1}{k}\sum_{i=1}^{k}x^r_{n+i}\right|^p < \infty$$

$$\Rightarrow x \in Z(p)$$

Therefore Z(p) is a complete p-normed space.

Theorem 2.2: Z β (p) = D(p) ∩ D_0

Proof: Let a ∈ D(p) ∩ D_0 and x ∈ Z(p).

Choose a natural number N > 1 such that

$$N > \max\left\{1,\ \sup_{k\geq 1 n\geq 0}\left|\frac{1}{k}\sum_{i=1}^{k}x_{n+i}\right|^p\right\}$$

Applying Abel's partial summation formula for a positive integer m, we have

$$\left|\sum_{k=1}^{m}a_k x_k\right| = \left|\sum_{k=1}^{m-1}(k\Delta a_k)\left(\frac{1}{k}\sum_{i=1}^{k}x_i\right) + ma_m\left(\frac{1}{m}\sum_{i=1}^{m}x_i\right)\right|$$

$$\leq \sup_{k\geq 1}\left|\frac{1}{k}\sum_{i=1}^{k}x_i\right|\sum_{k=1}^{m-1}|k\Delta a_k| + |ma_m|\sup_{m\geq 1}\left|\frac{1}{m}\sum_{i=1}^{m}x_i\right|$$

$$\leq \sup_{k\geq 1 n\geq 0}\left|\frac{1}{k}\sum_{i=1}^{k} x_{n+i}\right|\sum_{k=1}^{m-1}|k\Delta a_k| + |ma_m| \sup_{n\geq 1 n\geq 0}\left|\frac{1}{m}\sum_{i=1}^{m} x_{n+i}\right|$$

Taking m → ∞ and second term of the above relation vanishes because a ∈ D0 and $\frac{1}{m}\sum_{i=1}^{m} x_{n+i}$ is finite. Therefore

$$\left|\sum_{k=1}^{\infty} a_k x_k\right| \leq \left(\sum_{k=1}^{\infty}|k\Delta a_k|\right) N^{1/p} < \infty$$

Thus, a ∊ Z β (p).

Conversely, suppose that a ∊ Z β (p), a ∉ D(p) ∩ D_0, then either a ∉ D(p) or a ∉ D0.

Let a ∉ D(p) and a ∊ D0, then

$$\sum_{k=1}^{\infty} k\Delta a_k\, N^{1/p} = \infty \qquad for\ some\ N > 1$$

Define a sequence x = (x_k), where xk = $N^{1/p}$, k = 1, 2, 3, . . .

for some N > 1, then x ∊ Z(p). But

$$\sum_{k=1}^{\infty} a_k x_k = 0$$

This is a contradiction. Hence, a ∊ D(p).

Next, we suppose a ∉ D_0 but a ∊ D(p) then l = limk→∞ kak. Define x = (xk) by xk = (−1)k, then x ∈ Z(p), but

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Σ akxk = l Σ(−1)k, i.e. Σ akxk

does not converge which contradicts that a ∈ Z β (p). Hence, a ∈ D0.

This completes the proof.

III. CESARO SEQUENCE SPACE OF ABSOLUTE TYPE IN A GENERAL BANACH SPACE

Let X and Y be Banach space and B(X, Y) be the Banach space of all bounded linear operators from X into Y with the usual operator norm and p, p, p, . . . be a constant sequence of real numbers. We define the Cesaro null sequence spaces Ces(p, x)$_0$, Cesaro convergent sequence spaces Ces(p, x)$_c$ and Cesaro bounded sequence spaces Ces(p, x)$_\infty$ as

$$Ces(p,x)_0 = \left\{\bar{x} = (x_k) : x_k \in X \ such\ that\ \frac{1}{k}\sum_{i=1}^{k}\|x_i\|^p \to 0\ as\ k\to\infty\right\} 3.1$$

$$Ces(p,x)_c = \left\{\bar{x} = (x_k) : x_k \in X\ and\ \exists x \in X\ s.t.\ \left(\frac{1}{k}\sum_{i=1}^{k}\|x_i - x\|\right)^p \to 0\ as\ k\to\infty\right\} 3.2$$

$$Ces(p,x)_\infty = \left\{\bar{x} = (x_k) : x_k \in X\ s.t.\ \sup_{k\geq 1}\left(\frac{1}{k}\sum_{i=1}^{k}\|x_i\|\right)^p < \infty\right\} 3.3$$

Thus

Ces(p, x)0 ⊂ Ces(p, x)c ⊂ Ces(p, x)∞

For x ∈ Ces(p, x)∞ we define

$$\rho(x) = \sup_{k\geq 1}\left(\frac{1}{k}\sum_{i=1}^{k}\|x_i\|\right)^{p}$$

The above spaces are complete p-normed spaces which can be proved on similar pattern as theorem 2.1. However, the proof is omitted.

IV. α-DUAL OF SEQUENCE SPACES IN OPERATOR FORM

With the appearance of Robinson’s paper [8] in 1950, wherein he considered the action infinite matrices of operators form a Banach space on sequences of elements of that space, Ko¨the-Toeplitz duals are generalized in the operator form [4], we recall the definition of the α-dual of Banach space X valued sequence spaces E [4] as

$$E^{\alpha}(X) = \left\{(A_k) : A_k \in B(X,Y),\ \sum_{k}\|A_k x_k\| < \infty\ forall(x_k) \in E(X)\right\} 4.1$$

Theorem 4.1. For any p,

$$Ces^{\alpha}(p,x)_0 = Ces^{\alpha}(p,x)_c = Ces^{\alpha}(p,x)_{\alpha} = m_1(B(X,Y))$$

where

$$m_1(B(X,Y)) = \left\{(A_k) : (A_k) \in B(X,Y)\ and\ \sum_{k=1}^{\infty}\|A_k\| < \infty\right\} 4.2$$

Proof.

Since

$$Ces(p,x)_0 \subset Ces(p,x)_c \subset Ces(p,x)_{\infty}$$

$$\Rightarrow Ces^{\alpha}(p,x)_{\infty} \subset Ces^{\alpha}(p,x)_c \subset Ces^{\alpha}(p,x)_0$$

Therefore we show that

$$m_1(B(X,Y)) \subset Ces^{\alpha}(p,x)_{\infty} \quad and \quad Ces^{\alpha}(p,x)_0 \subset m_1(B(X,Y))$$

Let (Ak) ∈ m1(B(X, Y)). Then

$$\sum_{k=1}^{\infty}\|A_k x_k\| \leq \sum_{k=1}^{\infty}\|kA_k\|\left(\frac{1}{k}\sum_{i=1}^{k}\|x_i\|\right)$$

$$< N^{\frac{1}{p}}\sum_{k=1}^{\infty}k\|A_k\|$$

$$< \infty$$

$$\Rightarrow (A_k) \in Ces^{\alpha}(p,x)_{\infty} \Rightarrow m_1(B(X,Y)) \subset Ces^{\alpha}(p,x)_{\infty}$$

We suppose that

$$(A_k) \in Ces^{\alpha}(p,x)_0 \quad but \quad (A_k) \notin m_1(B(X,Y))$$

then there exists an increasing sequence n(i) such that

$$\sum_{n(i)+1}^{n(i+1)} k\|A_k\| > 2^i, \qquad \forall i \in N$$

and a sequence (zk) in S such that

$$2\|A_k z_k\| \geq \|A_k\|$$

Define

$$x_k = \frac{k z_k}{i\left[n(i+1)\right]}, \qquad for\ n(i) < k \leq n(i+1), \quad otherwise\ 0$$

Then (xk) $\in$ Ces(p, x)$_0$ and

$$\sum_{n(i)+1}^{n(i+1)} \|A_k x_k\| > \frac{1}{i[n(i+1)]} \sum_{n(i)+1}^{n(i+1)} \frac{1}{2} k\|A_k\| > \frac{1}{n(i+1)} > 1$$

which contradicts that (Ak) $\in$ Ces(p, x)$_0$. This completes the proof.

Theorem 4.2 :

$$m_1(B(X,Y)) \subset \ell_1(B(X,Y)),$$

where

$$\ell_1(B(X,Y)) = \left\{(A_k) : A_k \in B(X,Y), \sum_{k=1}^{\infty} \|A_k\| < \infty\right\}.$$

Proof : Let (Ak) $\in$ m$_1$(B(X, Y))

$$\Rightarrow \sum_{k=1}^{\infty} \|A_k\| \leq \sum_{k=1}^{\infty} k\|A_k\| < \infty$$

therefore

$$m_1(B(X,Y)) \subset \ell_1(B(X,Y)).$$

But the containment in the theorem 4.2 is strict. We construct an example to prove this assertion.
Let X∗ be continuous dual of X.
Take

$$f \in X^{*} with \|f\| = 1$$

and y $\in$Y, such that || y || = 1
Take a sequence (yk) such that $y_k \in Y$. Define

$$A_k \in B(X,Y) \quad foreach k \in N \quad as \quad A_k z = f(z) y_k, \quad foreach z \in X.$$

Now put

$$y_k = \frac{y}{k^2}$$

then

$$\|y_k\| = \frac{1}{k^2} \quad and \quad \|A_k\| = \|y_k\|.$$

So that

$$\sum_{k=1}^{\infty} \|A_k\| converges \quad but \quad \sum_{k=1}^{\infty} k\|A_k\| diverges.$$

Therefore the containment in the theorem 4.2 is strict and the proof is complete.

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