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Study on Properties of Certain Subclass of Univalent Functions with Negative Coefficients Related to Fractional Calculus Operator

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Abstract: *In this Paper, we have discussed a subclass* $TS(\gamma,\alpha,\mu,\lambda)$ *of univalent functions with negative coefficients related to fractional calclus operator in the unit disk* $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. *We obtain basic properties like coefficient inequality, distortion and covering theorem, radii of starlikeness, convexity and close-to-convexity, extreme points, Hadamard product, and closure theorems for functions belonging to our class.*

Keywords: Univalent, Fractional Operator, Starlike, Extreme points, Hadmard Product

I. INTRODUCTION

Let A denote the class of all functions $u(z)$ of the form

$$u(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1}$$

in the open unit disc $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0)=u'(0)-1=0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in $\mathbb{U}$. A function $u\in A$ is a starlike function of the order $m, 0\le m<1$, if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\}>m, z\in\mathbb{U}. \tag{2}$$

We denote this class with $S^*(m)$.

A function $u\in A$ is a convex function of the order $m, 0\le m<1$, if it satisfy

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\}>m, z\in\mathbb{U}. \tag{3}$$

We denote this class with $K(m)$.

Note that $S^*(0)=S^*$ and $K(0)=K$ are the usual classes of starlike and convex functions in $\mathbb{U}$ respectively.

Let T denote the class of functions analytic in $\mathbb{U}$ that are of the form

$$u(z)=z-\sum_{n=2}^{\infty}a_n z^n,\quad a_n\ge 0\ z\in\mathbb{U} \tag{4}$$

and let $T^*(m)=T\cap S^*(m)$, $C(m)=T\cap K(m)$. The class $T^*(m)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [14] .

Many essentially equivalent definitions of fractional calculus have been given in the literature [15] . We state the following definitions due to Owa and Srivastava which have been used rather frequently in the theory of analytic functions.

Definition 1.1: The fractional integral of order λ is defined, for a function (z) , by

$$D_z^{-\lambda}u(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}d\zeta,\ \ (\lambda>0) \tag{5}$$

and the fractional derivative of order μ is defined, for a function $u(z)$, by

$$D_z^{\lambda}u(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta,\ \ (0\le\lambda<1) \tag{6}$$

where$u(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ involved in (and that of $(z-\zeta)^{-\lambda}$ involved in is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

Definition 1.2: Under the hypotheses of Definition 6.1.1.the fractional derivative of order $n+\lambda$ is defined by

$$D_z^{n+\lambda}u(z)=\frac{d^n}{dz^n}D_z^{\lambda}u(z),\ \ (0\le\lambda<1;n\in\mathrm{N}_0=\mathrm{N}\cup\{0\}). \tag{7}$$

With the aid of the above definitions, Owa and Srivastava defined the fractional operator $\mathcal{J}_z^{\lambda}$ by

$$\mathcal{J}_z^{\lambda}u(z)=\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}u(z),\ \ (\lambda\neq 2,3,4,\cdots)$$

$$=z+\sum_{n=2}^{\infty}\phi\,(\lambda,n)a_nz^n$$

where

$$\phi(\lambda,n)=\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n-\lambda+1)} \tag{8}$$

And

$$\phi(\lambda,2)=\frac{2}{(2-\lambda)} \tag{9}$$

Now, by making use of the linear operator $\mathcal{J}_z^{\lambda}u$, we define a new subclass of functions belonging to the class A.

Now, we define a new subclass of functions belonging to the class A.

Definition 1.3: For $0\le\gamma<1, 0\le\alpha<1, 0<\mu<1$, and $0\le\lambda<1$, we let $TS(\gamma,\alpha,\mu,\lambda)$ be the subclass of u consisting of functions of the form (6.4) and its geometrical condition satisfy

$$\left|\frac{\gamma\left((\mathcal{J}_z^{\lambda}u(z))'-\frac{\mathcal{J}_z^{\lambda}u(z)}{z}\right)}{\alpha(\mathcal{J}_z^{\lambda}u(z))'+(1-\gamma)\frac{\mathcal{J}_z^{\lambda}u(z)}{z}}\right|<\mu,\ \ z\in\mathbb{U}$$

where$\mathcal{J}_z^{\lambda}$, is given by (6.8)

2. Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS(\gamma,\alpha,\mu,\lambda)$.

Theorem 2.1: Let the function u be defined by . Then $u \in TS(\gamma,\alpha,\mu,\lambda)$ if and only if

$\sum_{n=2}^{\infty}[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)a_n \le \mu(\alpha+(1-\gamma)),$ (10)

where$0<\mu<1, 0\le\gamma<1, 0\le\alpha<1,$ and $0\le\lambda<1$. The result (10)is sharp for the function

$$u(z)=z-\frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}z^n,\ \ n\ge 2.$$

Proof. Suppose that the inequality holds true and $|z|=1$. Then we obtain

$$\left|\gamma\left(\left(\mathcal{J}_z^{\lambda}u(z)\right)'-\frac{\mathcal{J}_z^{\lambda}u(z)}{z}\right)\right|-\mu\left|\alpha\left(\mathcal{J}_z^{\lambda}u(z))'+(1-\gamma)\frac{\mathcal{J}_z^{\lambda}u(z)}{z}\right)\right|$$

$$=\left|-\gamma\sum_{n=2}^{\infty}(n-1)\phi(n,\lambda)a_n z^{n-1}\right|$$

$$-\mu\left|\alpha+(1-\gamma)-\sum_{n=2}^{\infty}(n\alpha+1-\gamma)\phi(n,\lambda)a_n z^{n-1}\right|$$

$$\le\sum_{n=2}^{\infty}[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)a_n-\mu(\alpha+(1-\gamma))$$

$$\le 0$$

Hence, by maximum modulus principle,$u\in TS(\gamma,\alpha,\mu,\lambda)$. Now assume that $u\in TS(\gamma,\alpha,\mu,\lambda)$ so that

$$\left|\frac{\gamma\left((\mathcal{J}_z^{\lambda}u(z))'-\frac{\mathcal{J}_z^{\lambda}u(z)}{z}\right)}{\alpha(\mathcal{J}_z^{\lambda}u(z))'+(1-\gamma)\frac{\mathcal{J}_z^{\lambda}u(z)}{z}}\right|<\mu,\ \ z\in\mathbb{U}$$

Hence

$$\left|\gamma\left((\mathcal{J}_z^{\lambda}u(z))'-\frac{\mathcal{J}_z^{\lambda}u(z)}{z}\right)\right|<\mu\left|\alpha\left(\mathcal{J}_z^{\lambda}u(z))'+(1-\gamma)\frac{\mathcal{J}_z^{\lambda}u(z)}{z}\right)\right|.$$

Therefore, we get

$$\left|-\sum_{n=2}^{\infty}\gamma(n-1)\phi(n,\lambda)a_n z^{n-1}\right|$$
$$<\ \ \mu\left|\alpha+(1-\gamma)-\sum_{n=2}^{\infty}(n\alpha+1-\gamma)\phi(n,\lambda)a_n z^{n-1}\right|.$$

Thus

$$\sum_{n=2}^{\infty}[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)a_n\le\mu(\alpha+(1-\gamma))$$

and this completes the proof.

Corollary 2.1: Let the function $u\in TS(\gamma,\alpha,\mu,\lambda)$.Then

$$a_n\le\frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}z^n,\ \ n\ge 2. \quad (11)$$

3. Distortion and Covering Theorem:

We introduce the growth and distortion theorems for the functions in the class $TS(\gamma,\alpha,\mu,\lambda)$

Theorem 3.1: Let the function $u \in TS(\gamma,\alpha,\mu,\lambda)$. Then

$$|z| - \frac{\mu(\alpha+(1-\gamma))}{\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]}|z|^2$$

$$\leq |u(z)|$$

$$\leq \quad |z| + \frac{\mu(\alpha+(1-\gamma))}{\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]}|z|^2$$

The result is sharp and attained

$$u(z) = z - \frac{\mu(\alpha+(1-\gamma))}{\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]}z^2.$$

Proof.

$$|u(z)| = \left|z - \sum_{n=2}^{\infty} a_n z^n\right| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n$$

$$\leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n$$

By Theorem 2.1, we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)]\phi(n,\lambda)}. \tag{12}$$

Thus

$$|u(z)| \leq |z| + \frac{\mu(\alpha+(1-\gamma))}{\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]}|z|^2.$$

Also

$$\begin{aligned} |u(z)| \quad &\geq |z| - \sum_{n=2}^{\infty} a_n |z|^n \\ &\geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{\mu(\alpha+(1-\gamma))}{\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]}|z|^2. \end{aligned}$$

Theorem 3.2: .Let $u \in TS(\gamma,\alpha,\mu,\lambda)$. Then

$$1 - \frac{2\mu(\alpha+(1-\gamma))}{\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]}|z| \leq |u'(z)| \leq 1 + \frac{2\mu(\alpha+(1-\gamma))}{\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]}|z|$$

with equality for

$$u(z) = z - \frac{2\mu(\alpha+(1-\gamma))}{\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]}z^2.$$

Proof: Notice that

$$\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]\sum_{n=2}^{\infty} n\, a_n$$
$$\leq \sum_{n=2}^{\infty} n\,[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)a_n$$
$$\leq \mu(\alpha+(1-\gamma)),$$

from Theorem 3.1., Thus

$$|u'(z)| = \left|1-\sum_{n=2}^{\infty} n a_n z^{n-1}\right|$$

$$\leq 1+\sum_{n=2}^{\infty} n\, a_n |z|^{n-1}$$

$$\leq 1+|z|\sum_{n=2}^{\infty} n\, a_n$$

$$\leq 1+|z|\frac{2\mu(\alpha+(1-\gamma))}{\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]} \qquad (13)$$

On the other hand

$$|u'(z)| = \left|1-\sum_{n=2}^{\infty} n a_n z^{n-1}\right|$$

$$\geq 1-\sum_{n=2}^{\infty} n\, a_n |z|^{n-1}$$

$$\geq 1-|z|\sum_{n=2}^{\infty} n\, a_n$$

$$\geq 1-|z|\frac{2\mu(\alpha+(1-\gamma))}{\phi(2,\lambda)[\gamma+\mu(2\alpha+1-\gamma)]} \qquad (14)$$

Combining (13) and (14) , we get the result.

4. Radii of Starlikeness, Convexity and Close-to-Convexity:

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS(\gamma,\alpha,\mu,\lambda)$.

Theorem 4.1: .Let $u \in TS(\gamma,\alpha,\mu,\lambda)$. Then u is starlike in $|z| < R_1$ of order δ, $0 \leq \delta < 1$, where

$$R_1 = \inf_n \left\{\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{(n-\delta)\mu(\alpha+(1-\gamma))}\right\}^{\frac{1}{n-1}}, \quad n \geq 2. \qquad (15)$$

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Proof. u is starlike of order $\delta, 0 \le \delta < 1$ if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \delta.$$

Thus it is enough to show that

$$\left|\frac{zu'(z)}{u(z)} - 1\right| = \left|\frac{-\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} a_n z^{n-1}}\right| \le \frac{\sum_{n=2}^{\infty}(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

Thus

$$\left|\frac{zu'(z)}{u(z)} - 1\right| \le 1-\delta \;\; if \;\; \sum_{n=2}^{\infty}\frac{(n-\delta)}{(1-\delta)} a_n |z|^{n-1} \le 1. \qquad (16)$$

Hence by Theorem 2.1, (16) will be true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \le \frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{\mu(\alpha+(1-\gamma)}$$

or if

$$|z| \le \left[\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{(n-\delta)\mu(\alpha+(1-\gamma))}\right]^{\frac{1}{n-1}}, n \ge 2. \qquad (17)$$

The theorem follows easily from (17) .

Theorem 4.2: Let $u \in TS(\gamma,\alpha,\mu,\lambda)$. Then u is convex in $|z| < R_2$ of order $\delta, 0 \le \delta < 1$, where

$$R_2 = \inf_n \left\{\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{n(n-\delta)\mu(\alpha+(1-\gamma))}\right\}^{\frac{1}{n-1}}, \;\; n \ge 2. \qquad (18)$$

Proof: u is convex of order $\delta, 0 \le \delta < 1$ if

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \delta.$$

Thus it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} n\, a_n z^{n-1}}\right| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} n\, a_n |z|^{n-1}}.$$

Thus

$$\left|\frac{zu''(z)}{u'(z)}\right| \le 1-\delta \;\; if \;\; \sum_{n=2}^{\infty}\frac{n(n-\delta)}{(1-\delta)} a_n |z|^{n-1} \le 1. \qquad (19)$$

Hence by Theorem 2.1, (19) will be true if

$$\frac{n(n-\delta)}{1-\delta}|z|^{n-1} \le \frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{\mu(\alpha+(1-\gamma)}$$

or if

$$|z| \le \left[\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{n(n-\delta)\mu(\alpha+(1-\gamma))}\right]^{\frac{1}{n-1}}, n \ge 2. \qquad (20)$$

The theorem follows from (20).

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Theorem 4.3: Let $u \in TS(\gamma,\alpha,\mu,\lambda)$. Then u is close-to-convex in $|z| < R_3$ of order δ, $0 \le \delta < 1$, where

$$R_3 = \inf_n \left\{\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{n\mu(\alpha+(1-\gamma))}\right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{21}$$

Proof: u is close-to-convex of order $\delta, 0 \le \delta < 1$ if

$$\Re\{u'(z)\} > \delta.$$

Thus it is enough to show that

$$|u'(z) - 1| = \left|-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \le \sum_{n=2}^{\infty} n\, a_n |z|^{n-1}.$$

Thus

$$|u'(z) - 1| \le 1 - \delta \;\; if \;\; \sum_{n=2}^{\infty} \frac{n}{(1-\delta)} a_n |z|^{n-1} \le 1. \tag{22}$$

Hence by Theorem 2.1, (22) will be true if

$$\frac{n}{1-\delta}|z|^{n-1} \le \frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{\mu(\alpha+(1-\gamma)}$$

or if

$$|z| \le \left[\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{n\mu(\alpha+(1-\gamma))}\right]^{\frac{1}{n-1}}, n \ge 2. \tag{23}$$

The theorem follows from (23)

5. Extreme Points:

In the following theorem, we obtain extreme points for the class $TS(\gamma,\alpha,\mu,\lambda)$.

Theorem 5.1: Let $u_1(z) = z$ and

$$u_n(z) = z - \frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)} z^n, \;\; \text{for} \;\; n = 2,3,\cdots.$$

Then $u \in TS(\gamma,\alpha,\mu,\lambda)$ if and only if it can be expressed in the form

$$u(z) = \sum_{n=1}^{\infty} \theta_n\, u_n(z), \;\; where \;\; \theta_n \ge 0 \;\; and \;\; \sum_{n=1}^{\infty} \theta_n = 1.$$

Proof: Assume that $u(z) = \sum_{n=1}^{\infty} \theta_n\, u_n(z)$, hence we get

$$u(z) = z - \sum_{n=2}^{\infty} \frac{\mu(\alpha+(1-\gamma))\theta_n}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)} z^n.$$

Now, $u \in TS(\gamma,\alpha,\mu,\lambda)$, since

$$\sum_{n=2}^{\infty} \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))} \times \frac{\mu(\alpha+(1-\gamma))\theta_n}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}$$

$$= \sum_{n=2}^{\infty} \theta_n = 1 - \theta_1 \le 1$$

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Conversely, suppose $u \in TS(\gamma,\alpha,\mu,\lambda)$. Then we show that u can be written in the form $\sum_{n=1}^{\infty}\theta_n u_n(z)$. Now $u \in TS(\gamma,\alpha,\mu,\lambda)$ implies from Theorem 2.1.

$$a_n \leq \frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}.$$

Setting $\theta_n = \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}a_n, n = 2,3,\cdots$

and $\theta_1 = 1-\sum_{n=2}^{\infty}\theta_n$, we obtain $u(z) = \sum_{n=1}^{\infty}\theta_n u_n(z)$.

6. Hadamard product:

In the following theorem, we obtain the convolution result for functions belongs to the class $TS(\gamma,\alpha,\mu,\lambda)$.

Theorem 6.1: Let $u, g \in TS(\gamma,\alpha,\mu,\lambda)$. Then $u * g \in TS(\gamma,\alpha,\zeta,\lambda)$ for

$$u(z) = z-\sum_{n=2}^{\infty}a_n z^n, g(z) = z-\sum_{n=2}^{\infty}b_n z^n \text{ and } (u*g)(z) = z-\sum_{n=2}^{\infty}a_n b_n z^n,$$

where

$$\zeta \geq \frac{\mu^2(\alpha+(1-\gamma))\gamma(n-1)}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2\phi(n,\lambda)-\mu^2(\alpha+(1-\gamma))(n\alpha+1-\gamma)}.$$

Proof:

$u \in TS(\gamma,\alpha,\mu,\lambda)$and so

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}a_n \leq 1, \quad (24)$$

and

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}b_n \leq 1. \quad (25)$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\zeta(n\alpha+1-\gamma)]\phi(n,\lambda)}{\zeta(\alpha+(1-\gamma))}a_n b_n \leq 1. \quad (26)$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}\sqrt{a_n b_n} \leq 1. \quad (27)$$

Therefore it is enough to show that

$$\frac{[\gamma(n-1)+\zeta(n\alpha+1-\gamma)]\phi(n,\lambda)}{\zeta(\alpha+(1-\gamma))}a_n b_n \leq \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}\sqrt{a_n b_n}.$$

That is

$$\sqrt{a_n b_n} \leq \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\zeta}{[\gamma(n-1)+\zeta(n\alpha+1-\gamma)]\mu}. \quad (28)$$

From

$$\sqrt{a_n b_n} \leq \frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}.$$

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Thus it is enough to show that

$$\frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}\le\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\zeta}{[\gamma(n-1)+\zeta(n\alpha+1-\gamma)]\mu},$$

which simplifies to

$$\zeta\ge\frac{\mu^2(\alpha+(1-\gamma))\gamma(n-1)}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2\phi(n,\lambda)-\mu^2(\alpha+(1-\gamma))(n\alpha+1-\gamma)}.$$

7. Closure Theorems:

We shall prove the following closure theorems for the class $TS(\gamma,\alpha,\mu,\lambda)$.

Theorem 7.1: Let u_j in TS(γ, α, μ, λ). j=1,2,…. Then

$$g(z)=\sum_{j=1}^{s}c_j\,u_j(z)\in TS(\gamma,\alpha,\mu,\lambda)$$

For $u_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}\,z^n$, where $\sum_{j=1}^{s}c_j=1$.

Proof.

$$\begin{aligned}g(z)&=\sum_{j=1}^{s}c_j\,u_j(z)\\&=z-\sum_{n=2}^{\infty}\sum_{j=1}^{s}c_j\,a_{n,j}z^n\\&=z-\sum_{n=2}^{\infty}e_n\,z^n,\end{aligned}$$

where $e_n=\sum_{j=1}^{s}c_j\,a_{n,j}$. Thus $g(z)\in TS(\gamma,\alpha,\mu,\lambda)$ if

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}e_n\le 1,$$

that is, if

$$\begin{aligned}&\sum_{n=2}^{\infty}\sum_{j=1}^{s}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}c_ja_{n,j}\\=&\sum_{j=1}^{s}c_j\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}a_{n,j}\\\le&\sum_{j=1}^{s}c_j=1.\end{aligned}$$

Theorem 7.2 : Let $u,g\in TS(\gamma,\alpha,\mu,\lambda)$. Then

$$h(z)=z-\sum_{n=2}^{\infty}(a_n^2+b_n^2)z^n\in TS(\gamma,\alpha,\mu,\lambda), where$$

$$\zeta\ge\frac{2\gamma(n-1)\mu^2(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2\phi(n,\lambda)-2\mu^2(\alpha+(1-\gamma))(n\alpha+1-\gamma)}.$$

Proof: Since $u, g \in TS(\gamma, \alpha, \mu, \lambda)$, so Theorem6.1.1.yields

$$\sum_{n=2}^{\infty}\left[\frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}a_n\right]^2 \leq 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}b_n\right]^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}\right]^2(a_n^2+b_n^2) \leq 1. \quad (29)$$

But $h(z) \in TS(\gamma, \alpha, \zeta, q, m)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\zeta(n\alpha+1-\gamma)]\phi(n,\lambda)}{\zeta(\alpha+(1-\gamma))}(a_n^2+b_n^2) \leq 1, \quad (30)$$

where $0 < \zeta < 1$, however implies (30) if

$$\frac{[\gamma(n-1)+\zeta(n\alpha+1-\gamma)]\phi(n,\lambda)}{\zeta(\alpha+(1-\gamma))} \leq \frac{1}{2}\left[\frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))\phi(n,\lambda)}{\mu(\alpha+(1-\gamma))}\right]^2.$$

Simplifying, we get

$$\zeta \geq \frac{2\gamma(n-1)\mu^2(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2\phi(n,\lambda)-2\mu^2(\alpha+(1-\gamma))(n\alpha+1-\gamma)}.$$

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